A FIXED POINT THEOREM

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Suppose that space is metric. A *chain* is a finite collection of open sets $d_1, d_2, \cdots, d_n$ such that d_i intersects d_j if and only if $|i-j| \leqq 1$. If the elements of a chain are of diameter less than a positive number ϵ, that chain is said to be an *ϵ-chain*. A compact continuum is said to be *chainable* if for each positive number ϵ, there is an ϵ-chain covering it. R. H. Bing has called [1] such continua *snake-like*. In 1951 O. H. Hamilton showed [4] that every compact chainable continuum has the fixed point property; i.e., that if f is a continuous mapping of such a continuum M into itself, then some point of M is its own image under f.

In the present paper it is shown that the Cartesian product of finitely many compact chainable continua has the fixed point property. Since arcs are compact chainable continua, this is a generalization of the Brouwer fixed point theorem. Two other examples of compact chainable continua are the closure of the graph of sin $(1/x)$, $0 < x \leqq 1$, and the pseudo-arc. Other examples are given in [1].

After reading the original manuscript of this paper, A. D. Wallace raised the question as to whether the Cartesian product of finitely many compact chainable continua is a quasi-complex (p. 323 of [6]). Rather surprisingly, the answer to this question is in the affirmative. A proof of this theorem is also given here. Since the Cartesian product of finitely many compact chainable continua is zero-cyclic, the fact that such products have the fixed point property is a special case of the Lefschetz fixed point theorem for zero-cyclic quasi-complexes. Since the author's original argument is of a very different nature, it is also given.

Let E^n denote Euclidean n-space and R^n the set of all points of E^n whose distance from the origin is not greater than one. Let S^{n-1} denote the set of all points of E^n whose distance from the origin is one. Let I denote the set of all points on the x-axis having abscissa x such that $0 \leqq x \leqq 1$, and let I^n denote the set of all points of E^n each of whose n coordinates x_i satisfies $0 \leqq x_i \leqq 1$. If P is a point of E^n having coordinates $(x_1, x_2, \cdots, x_n)$ and t is a real number, by tP is meant the point of E^n having coordinates $(tx_1, tx_2, \cdots, tx_n)$.

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For definitions of those concepts from homotopy theory which are used in this paper the reader is referred to §3 of Chapter VI of [5].

LEMMA 1. *Suppose f and g are continuous mappings of R^n into itself and $g|S^{n-1}$ is an essential mapping onto S^{n-1}. Then there is a point x in R^n such that $f(x)=g(x)$.*

PROOF. Suppose there is no point x in R^n such that $f(x)=g(x)$. For each point x in R^n let $h(x)=g(x)-f(x)$. For each point x in S^{n-1} and number t, $0\leqq t\leqq 1$, let $H(x, t)=h(tx)/|h(tx)|$. This is possible since $h(x)\neq 0$ for any x in R^n. $H(x, t)$ is continuous in x and t.

If $H(x, 1)$ and $g(x)$ were not antipodal for any point x in S^{n-1}, they would be homotopic; but $g|S^{n-1}$ is essential and $H(S^{n-1}, 1)$ is homotopic to the constant map $H(S^{n-1}, 0)$. Hence, there is a point x in S^{n-1} such that $H(x, 1)=-g(x)$. Then $g(x)\cdot(1+|g(x)-f(x)|)=f(x)$. Let $k=1+|g(x)-f(x)|$. $k\cdot g(x)=f(x)$. Since $|g(x)|=1$ and $|f(x)|\leqq 1$, $k\leqq 1$; but by its definition $k\geqq 1$. Therefore, $g(x)=f(x)$.

LEMMA 2. *Suppose that for each integer i, $1\leqq i\leqq n$, f_i is a continuous mapping of I onto I such that $f_i(0)=0$ and $f_i(1)=1$. For each point $x=(x_1, x_2, \cdots, x_n)$ of I^n let $f(x)=(f_1(x_1), f_2(x_2), \cdots, f_n(x_n))$. Let T^{n-1} denote the set of all points x of I^n such that for some i, x_i is either 0 or 1. Then $f(T^{n-1})=T^{n-1}$, and $f|T^{n-1}$ is homotopic to the identity map of T^{n-1} onto itself.*

PROOF. For each integer i, $1\leqq i\leqq n$, there is a continuous mapping $h_i(x, t)$ of $I\times I$ onto I such that $h_i(x, 0)=f_i(x)$, $h_i(x, 1)=x$, and $h_i(0, t)=0$ and $h_i(1, t)=1$ for $0\leqq t\leqq 1$. Let $H(x, t)$, $0\leqq t\leqq 1$ and x in I^n, be defined as follows: $H(x, t)=(h_1(x_1, 1), \cdots, h_{i-1}(x_{i-1}, 1), h_i(x_i, u), h_{i+1}(x_{i+1}, 0), \cdots, h_n(x_n, 0))$, where $(i-1)/n\leqq t\leqq i/n$ and $u=n\cdot[t-(i-1)/n]$. It can easily be shown that $H(x, t)$ is continuous in x and t, that $H(x, 0)=f(x)$ for all x in I^n, that $H(x, 1)=x$ for all x in T^{n-1} (in fact in I^n), and that $H(T^{n-1}, t)=T^{n-1}$ for all t, $0\leqq t\leqq 1$.

THEOREM 1. *Suppose that M is the Cartesian product of n compact chainable continua $X_1, X_2, \cdots, X_n$ and f is a continuous mapping of M into itself. Then there is a point x of M such that $x=f(x)$.*

PROOF. Let M be all of space. Suppose there is no point x of M such that $x=f(x)$. There is a positive number e such that $d(x, f(x))>4ne$ for all points x in M. For each integer i, $1\leqq i\leqq n$, there is a chain A_i covering X_i such that each element of the collection C_a of all ordered n-tuples $(a_1, a_2, \cdots, a_n)$, where a_i is in A_i, has diameter less than e. For each integer i, $1\leqq i\leqq n$, there is a chain B_i' covering

X_i such that B_i' is a refinement of A_i and each element of the collection C_b' of all ordered n-tuples $(b_1, b_2, \cdots, b_n)$, where b_i is in B_i', is mapped by f into some element of C_a. For each i let B_i denote a subchain of B_i' which is irreducible with respect to covering points in the first and last links of A_i which are not in any other link of A_i, and let C_b denote the subcollection of C_b' of all ordered n-tuples $(b_1, b_2, \cdots, b_n)$, where b_i is in B_i. For each i let α_i denote the number of links of A_i and β_i denote the number of links of B_i. If for some i, the first link of B_i lies in the last link of A_i, renumber the links of B_i so that its jth link becomes its $(\beta_i - j + 1)$st link.

For each $x = (x_1, x_2, \cdots, x_n)$, where $x_i = k_i/(\beta_i - 1)$, k_i being an integer such that $0 \leqq k_i \leqq \beta_i - 1$, let ρ_x be the element of C_b whose ith term is the (k_i+1)st link in B_i. $f(\rho_x)$ is in some element θ of C_a. The ith term of θ is the p_ith link of A_i. There are two adjacent positive integers m_i and m_i+1 such that the ith term of any element of C_a containing $f(\rho_x)$ is either the m_ith or the (m_i+1)st link of A_i and one of the numbers m_i and m_i+1 is p_i. If no element of C_a containing $f(\rho_x)$ has an ith term other than the p_ith link of A_i, let

$$f_i(x) = (p_i - 1)/(\alpha_i - 1).$$

If some element of C_a containing $f(\rho_x)$ has an ith term other than the p_ith link of A_i, let $f_i(x) = (2m_i - 1)/2(\alpha_i - 1)$. Let

$$F(x) = (f_1(x), f_2(x), \cdots, f_n(x)).$$

Let B denote the set of all points $x = (x_1, x_2, \cdots, x_n)$ of I^n such that for each integer i, $1 \leqq i \leqq n$, there is an integer k_i, $0 \leqq k_i \leqq \beta_i - 1$, such that $x_i = k_i/(\beta_i - 1)$. Let $\mathfrak{B}$ denote the collection of all sets σ of 2^n points of B for which there is a point P of σ such that for every point Q of σ and for each integer i, $1 \leqq i \leqq n$, the ith coordinate of Q either equals the ith coordinate of P or exceeds it by 1. It can be shown (see, for example, §8 of Chapter II of [3]) that there is an n-complex β which is a triangulation of I^n such that the vertex set of each simplex in β is a subset of an element of $\mathfrak{B}$.

For each point x in I^n and simplex σ of β containing it, let $(a_0, a_1, \cdots, a_j)$ be the barycentric coordinates of x with respect to the vertex set $(s_0, s_1, \cdots, s_j)$ of σ. $x = a_0 s_0 + a_1 s_1 + \cdots + a_j s_j$. Define $F_\sigma(x)$ to be $a_0 F(s_0) + a_1 F(s_1) + \cdots + a_j F(s_j)$. Clearly, F_σ is continuous on σ. If x is common to two simplexes σ_1 and σ_2 of β, let σ denote their common face. Then $F_{\sigma_1}(x) = F_\sigma(x) = F_{\sigma_2}(x)$. Hence, if $F(x)$ denotes $F_\sigma(x)$ for any simplex σ of β containing x, F is a continuous mapping of I^n into itself. Furthermore, if P is a vertex of a simplex of β containing the point x, for each integer i, $1 \leqq i \leqq n$, the

ith coordinates of $F(P)$ and $F(x)$ do not differ by more than $1/(\alpha_i-1)$. Let $g_i((k-1)/(\beta_i-1))=(j-1)/(\alpha_i-1)$ if the kth link of B_i lies in only the jth link of A_i. If the kth link of B_i lies in two links of A_i, the jth and $(j+1)$st, let $g_i((k-1)/(\beta_i-1))=(2j-1)/2(\alpha_i-1)$. Let $g_i(I)=I$ be the piecewise linear extension of this mapping. For each point x of I such that $(k-1)/(\beta_i-1)\leqq x\leqq k/(\beta_i-1)$, $g_i(x)$ is between $g_i((k-1)/(\beta_i-1))$ and $g_i(k/(\beta_i-1))$. For each point x in I^n, let $g(x)=(g_1(x_1), g_2(x_2), \cdots, g_n(x_n))$. Let T^{n-1} denote the boundary of I^n. By Lemma 2, $g|T^{n-1}$ is homotopic to the identity mapping of T^{n-1} onto itself; hence, $g|T^{n-1}$ is essential onto T^{n-1}.

By Lemma 1, there is a point x of I^n such that $F(x)=g(x)$. Let v denote a vertex of a simplex of β containing x. For each integer i, $1\leqq i\leqq n$, the ith coordinate of $F(x)$ does not differ from the ith coordinate of $F(v)$ by more than $1/(\alpha_i-1)$. Also, the ith coordinate of $g(x)$ does not differ from the ith coordinate of $g(v)$ by more than $1/(\alpha_i-1)$. Hence, the ith coordinate of $F(v)$ does not differ from the ith coordinate of $g(v)$ by more than $2/(\alpha_i-1)$. The point v is an element of the set B; i.e., $v=(v_1, v_2, \cdots, v_n)$, where for each integer i, $1\leqq i\leqq n$, there is an integer k_i, $0\leqq k_i\leqq\beta_i-1$, such that $v_i=k_i/(\beta_i-1)$. Let V denote the element of the set C_b whose ith term is the (k_i+1)st link of B_i. Let $\mathfrak{G}$ denote an element of C_a such that for each i, the ith term of V lies in the ith term of $\mathfrak{G}$ and let $\mathfrak{F}$ be an element of C_a containing $f(V)$. For each i there is a subchain in A_i from the ith term of $\mathfrak{G}$ to the ith term of $\mathfrak{F}$ having not more than 4 links. Therefore, there is a chain in C_a from $\mathfrak{G}$ to $\mathfrak{F}$ having not more than $4n$ links. Each of these links is of diameter less than e; therefore, $d(V, f(V)) < 4ne$. This is a contradiction.

COROLLARY. *The Cartesian product of the elements of any collection of compact chainable continua has the fixed point property.*

PROOF. This follows immediately from Theorem 1 and the theorem that if G is a collection of compact Hausdorff spaces such that the Cartesian product of the elements of each finite subcollection of G has the fixed point property, then the Cartesian product of the elements of the collection G has the fixed point property.

For the definition of the term *quasi-complex* the reader is referred to p. 323 of [6]. Notation and terminology used, with only a few exceptions, are in conformity with that used in [6].

LEMMA 3. *If α and β are arc-like finite simplicial complexes and π is a simplicial mapping of β onto α, there exists a chain mapping ω of α into β such that*

(i) *for each zero-chain γ^0 of α, $KI(\gamma^0)=KI(\omega(\gamma^0))$;*

(ii) *for each i-simplex* σ^i *of* α *and i-simplex* γ^i *of* β *having nonzero coefficient in* $\omega(1\cdot\sigma^i)$, $\pi(\gamma^i)\subset\sigma^i$; *and* (iii) $\omega\pi\sim 1$.

PROOF. Let $a_1, a_2, \cdots, a_n$ denote the vertices of α ordered as on α. There is a subarc β' of β such that $\pi(\beta')=\alpha$ and there is no proper subarc γ of β' such that $\pi(\gamma)=\alpha$. Let b_1 denote the vertex of β' such that $\pi(b_1)=a_1$ and let $b_1, b_2, \cdots, b_m$ denote the vertices of β' ordered as on β'. There is a subsequence $b_{k_1}, b_{k_2}, \cdots, b_{k_p}$ of $b_1, b_2, \cdots, b_m$ such that

(1) $\pi(b_{k_1})=a_1$ and $\pi(b_{k_p})=a_n$;
(2) if $\pi(b_{k_i})=a_j$ and $\pi(b_{k_{i+1}})=a_k$, then $|j-k|\leqq 1$; and
(3) for each i, k_{i+1} is the greatest integer j such that
(a) $k_i\leqq j\leqq k_p$ and
(b) if $k_i<q\leqq j$, $\pi(b_q)\in\{\pi(b_{k_i})\}\cup\{\pi(b_{k_i+1})\}$.

Define $\omega(1\cdot a_i)$ to be $\sum_{j=1}^{p} X_i^j\cdot b_{k_j}$, where $X_i^j=0$ if $\pi(b_{k_j})\neq a_i$,

$$\text{and if } \pi(b_{k_j})=a_i \text{ and } \begin{cases} \pi(b_{k_j+1}) = a_{i+1}, & \text{then } X_i^j = +1.\\ \pi(b_{k_j+1}) = a_{i-1}, & \text{then } X_i^j = -1.\\ \qquad j = p, & \text{then } X_j^j = +1.\end{cases}$$

Define $\omega(1\cdot(a_i, a_{i+1}))=\sum_{j=1}^{m-1} Y_i^j\cdot(b_j, b_{j+1})$, where if $k_s\leqq j<j+1\leqq k_{s+1}$ and for all t, $k_s\leqq t\leqq k_{s+1}$, $\pi(b_t)\in\{a_i\}\cup\{a_{i+1}\}$, then $Y_i^j=+1$; otherwise, $Y_i^j=0$.

To show that ω is a chain mapping, it will be shown that for each j, the coefficient of b_j is the same in both $\omega\delta(1\cdot(a_i, a_{i+1}))$ and $\delta\omega(1\cdot(a_i, a_{i+1}))$, where δ denotes the boundary operator. Unless j is some k_t, its coefficient in both expressions is zero. Suppose $j=k_t$.

CASE 1. $\pi(b_{k_t})=a_{i+1}$. If $\pi(b_{k_t+1})=a_{i+2}$, then $\{\pi(b_{k_{t-1}})\}\cup\{\pi(b_{k_{t-1}+1})\}\subset\{a_i\}\cup a_{i+1}\}$ and $\omega(a_i, a_{i+1})$ contains $1\cdot(b_{k-1}, b_{k_t})$ and $0\cdot(b_{k_t}, b_{k_t+1})$. If $\pi(b_{k_t+1})=a_i$, then either $\pi(b_{k_{t-1}})$ or $\pi(b_{k_{t-1}+1})$ does not lie in $\{a_i\}\cup\{a_{i+1}\}$ and $\omega(a_i, a_{i+1})$ contains $0\cdot(b_{k_t-1}, b_{k_t})$ and $1\cdot(b_{k_t}, b_{k_t+1})$. Thus, $\delta\omega(a_i, a_{i+1})$ contains $+1\cdot b_{k_t}$ (or $-1\cdot b_{k_t}$) if $\pi(b_{k_t+1})=a_{i+2}$ (or $=a_i$). This is the same coefficient for b_{k_t} as that given by $\omega(1\cdot a_{i+1})$, which is the same as that given by $\omega\delta(a_i, a_{i+1})$.

CASE 2. $\pi(b_{k_t})=a_i$. If $\pi(b_{k_t+1})=a_{i+1}$, then $\omega(a_i, a_{i+1})$ contains $0\cdot(b_{k_t-1}, b_{k_t})$ and $1\cdot(b_{k_t}, b_{k_t+1})$. If $\pi(b_{k_t+1})=a_{i-1}$, then $\omega(a_i, a_{i+1})$ contains $1\cdot(b_{k_t-1}, b_{k_t})$ and $0\cdot(b_{k_t}, b_{k_t+1})$. Thus, $\delta\omega(a_i, a_{i+1})$ contains $-1\cdot b_{k_t}$ (or $+1\cdot b_{k_t}$) if $\pi(b_{k_t+1})=a_{i+1}$ (or $=a_{i-1}$). This is the same coefficient for b_{k_t} as that given by $\omega(-1\cdot a_i)$, which is the same as that given by $\omega\delta(a_i, a_{i+1})$. Thus, ω is a chain mapping.

To show that ω preserves the Kronecker index of zero-cycles, it will be shown that $\mathrm{KI}(\omega(1\cdot a_i))=1$, for each i. Clearly, $\mathrm{KI}(\omega(1\cdot a_1))$

$=\mathrm{KI}(\omega(1\cdot a_n))=1$. Suppose $1<i<n$. Let U_i be the collection of all subarcs γ of β' which are maximal with respect to $\pi(\gamma)=a_i$ and such that if $b_j, b_{j+1}, \cdots, b_{j+k}$ are the vertices of γ, $\pi(b_{j-1})=a_{i-1}$ and $\pi(b_{j+k+1})=a_{i+1}$. Let D_i be the collection of all such maximal subarcs for which $\pi(b_{j-1})=a_{i+1}$ and $\pi(b_{j+k+1})=a_{i-1}$. For each element of $U_i\cup D_i$, the vertex b_{j+k} is a b_{k_t}, and $\omega(1\cdot a_i)$ contains $+1\cdot b_{j+k}$ or $-1\cdot b_{j+k}$ as b_{j+k} is a vertex of an element of U_i or D_i. Also $\omega(1\cdot a_i)$ does not attach a nonzero coefficient to any b_{k_t} which is not a vertex of some element of $U_i\cup D_i$. Thus, $\mathrm{KI}(\omega(1\cdot a_i))$ equals the number of elements of U_i minus the number of elements of D_i. If the elements of $U_i\cup D_i$ are ordered as on β', between each two elements of U_i there is an element of D_i and between each two elements of D_i there is an element of U_i; furthermore, the first and last elements of $U_i\cup D_i$ are in U_i. Hence, U_i has one more element than D_i, and $\mathrm{KI}(\omega(1\cdot a_i))=1$.

For cycles γ^p on β of dimension $p=1$, $\omega\pi(\gamma^p)$ is a cycle; also $\omega\pi(\gamma^p)-\gamma^p$ is a cycle. Since β is zero-cyclic, $\omega\pi(\gamma^p)-\gamma^p\sim 0$ or $\omega\pi(\gamma^p)\sim\gamma^p$. For any zero-cycle, γ^0, on β, $\mathrm{KI}(\omega\pi(\gamma^0))=\mathrm{KI}(\gamma^0)$, and so $\omega\pi(\gamma^0)\sim\gamma^0$. Hence, $\psi\pi\sim 1$.

THEOREM 2. *Every compact metric chainable continuum is a quasi-complex.*

PROOF. Let M denote a compact metric chainable continuum. Let $U_1, U_2, \cdots$ be a sequence of chains covering M such that U_{i+1} is a refinement of U_i and U_i is a $(1/i)$-chain; let Φ_i denote the nerve of U_i. If i and j are positive integers and $i<j$, let π_i^j denote a simplicial mapping of Φ_j onto Φ_i induced by inclusion; i.e., a projection mapping. Let ω_j^i denote the chain mapping of Φ_i into Φ_j defined for π_i^j as in Lemma 3. Antiprojections will be the mappings ω_j^i and finite products $\omega_{i_n}^{i_{n-1}}\cdots\omega_{i_3}^{i_2}\cdot\omega_{i_2}^{i_1}$, where $i_1<i_2<\cdots<i_n$. These products also preserve the Kronecker index of zero-cycles on Φ_{i_1}. These mappings have all of the properties required in Property B on p. 323 of [6]. To show that M has Property C, for each $U_i(=a)$, let $U_j(=g)$ be a sufficiently small refinement of U_i that any three adjacent elements of U_j lie in some element of U_i. Then for any $U_k(=b)$, let $U_m(=h)$ be the one of the two U_j and U_k which is a refinement of both. Therefore, M has Property C.

For two complexes K_1 and K_2, $K_1\Delta K_2$ denotes the simplicial product of K_1 and K_2 as defined in §8 of Chapter II of [3].

LEMMA 4. *If for each i, $1\leq i\leq n$, α_i is a finite simplicial complex, β_i is a connected finite simplicial complex, and ω_i maps the zero-chains of α_i into zero-chains of β_i in such a way that* $\mathrm{KI}(\omega_i\delta(\sigma^1))=0$ *for each one-simplex σ^1 of α_i, then if $\alpha=\alpha_1\Delta\alpha_2\Delta\cdots\Delta\alpha_n$, $\beta=\beta_1\Delta\beta_2\Delta\cdots$*

$\Delta\beta_n$, and for each vertex $v=(a_1, a_2, \cdots, a_n)$ of α, where a_i is a vertex of α_i, $\omega(1\cdot v)=\sum\gamma_b\cdot(b_1, b_2, \cdots, b_n)$, where the summation extends over all vertices $b=(b_1, b_2, \cdots, b_n)$ of β, and for each b, γ_b is the product of the coefficients of the b_i in $\omega_i(a_i)$, then if γ^0 is a bounding zero-cycle in α, $\omega(\gamma^0)$ is a bounding zero-cycle in β.

PROOF. For each vertex v of α, the Kronecker index of $\omega(v)$ equals the product of the Kronecker indices of the $\omega_i(a_i)$, where $\{a_i\}$ are the coordinates of v. If γ^0 is the boundary of the one-simplex (v_1, v_2) of α, where all of the coordinates of v_1 and v_2 are the same except the ith, then $\mathrm{KI}(\omega(\gamma^0))=\mathrm{KI}(\omega(v_2))-\mathrm{KI}(\omega(v_1))=0$, since $\mathrm{KI}(\omega_i\delta(v_{1i}, v_{2i}))=0$. Since β is connected, $\omega(\gamma^0)$ bounds. If γ^0 is the boundary of any one-simplex in α, it is the sum of the boundaries of one-simplexes in α of the sort discussed in the previous sentence; hence, $\mathrm{KI}(\omega(\gamma^0))=0$. Finally, if $\gamma^0=\delta\sum_{k=1}^{j} p_k\cdot\sigma_k^1$, where each σ_k^1 is a one-simplex of α and p_k is an element of the coefficient group, $\omega(\gamma^0)=\omega(\sum_{k=1}^{j} p_k\cdot\delta(\sigma_k^1))=\sum_{k=1}^{j} p_k\cdot\omega\delta(\sigma_k^1)$; therefore, $\mathrm{KI}(\omega(\gamma^0))=0$. Hence, in each case $\omega(\gamma^0)$ bounds if γ^0 bounds.

LEMMA 5. *If σ is a complex, σ' is a subcomplex of σ containing all of the vertices of σ, γ is a zero-cyclic complex, and ω' is a chain mapping of σ' into γ which maps bounding zero-cycles on σ' into bounding zero-cycles on γ, then ω' can be extended to a chain mapping ω of σ into γ.*

PROOF. Let $\alpha_1, \alpha_2, \cdots, \alpha_k$ denote the simplexes in σ which are not in σ' ordered such that if $i<j$, then the dimension of α_i is less than or equal to the dimension of α_j. The argument will be by mathematical induction. For each i, $1\leqq i\leqq k$, let σ_i denote the subcomplex of σ consisting of all simplexes of σ' and all simplexes in each of the simplexes $\alpha_1, \alpha_2, \cdots, \alpha_i$. ω_1 is the extension of ω' to σ_1 obtained as follows: since $\delta(\alpha_1)$ is in σ' and is a bounding cycle, $\omega'\delta(\alpha_1)$ is a cycle in γ which bounds in γ if it is a zero-cycle since it is the image of a bounding zero-cycle and bounds in γ if it is a higher-dimensional cycle since all such cycles bound in γ. Let τ_1 denote a chain in γ such that $\omega'\delta(\alpha_1)=\delta\tau_1$ and let $\omega_1(\alpha_1)=\tau_1$. Then ω_1 is a chain mapping of σ_1 into γ and is an extension of ω'. It is clear this process can be continued inductively.

An *n-product simplex* is a simplex which is the Δ product of n simplexes.

LEMMA 6. *If for each i, $1\leqq i\leqq n$, α_i and β_i are arc-like finite simplicial complexes and π_i is a simplicial mapping of β_i onto α_i, then if for each vertex b of $\beta=\beta_1\Delta\beta_2\Delta\cdots\Delta\beta_n$, $\pi(b)$ is the vertex $(\pi_1(b_1), \pi_2(b_2), \cdots, \pi_n(b_n))$ of $\alpha=\alpha_1\Delta\alpha_2\Delta\cdots\Delta\alpha_n$, where for each i, b_i is the ith coordinate* *of b, then π is a simplicial mapping of β onto α and there is a chain mapping ω of α into β such that*

(i) *for each zero-chain γ^0 of α,* $\mathrm{KI}(\gamma^0)=\mathrm{KI}(\omega(\gamma^0))$;

(ii) *for each k-simplex σ^k of α, (2^n-1)-simplex σ of α of which σ^k is a face, and k-simplex γ^k of β having nonzero coefficient in* $\omega(1\cdot\sigma^k)$, $\pi(\gamma^k)\subset\sigma$; *and*

(iii) $\omega\pi\sim 1$.

Proof. For each vertex $v=(a_1, a_2, \cdots, a_n)$ of α and vertex $u=(b_1, b_2, \cdots, b_n)$ of β, let $\omega_u(v)=\lambda_u\cdot u$, where λ_u is the product of the coefficients of the b_i in $\omega_i(a_i)$, ω_i being defined for α_i, β_i, and π_i as in Lemma 3. Let $\omega'(v)$ be $\sum\omega_u(v)$, where the summation is over all vertices u in β. $\mathrm{KI}(\omega'(v))$ is the product of the Kronecker indices of the $\omega_i(a_i)$ and since each of these is one, $\mathrm{KI}(\omega'(v))=1$.

Let $\gamma_1, \gamma_2, \cdots, \gamma_k$ denote the n-product simplexes of α of positive dimension ordered so that if $i<j$, then the dimension of γ_i is less than or equal to the dimension of γ_j. For each j, let ρ_j denote the subcomplex of α composed of all vertices of α and all faces of all simplexes $\gamma_1, \gamma_2, \cdots, \gamma_j$. $\gamma_1=A_1\Delta A_2\Delta\cdots\Delta A_n$, where each A_i is a simplex in α_i. Let B_i denote the collection of all maximal coherent subcomplexes of β_i each simplex of which has a nonzero coefficient in $\omega_i(A_i)$. For each subcomplex $\mu=X_1\Delta X_2\Delta\cdots\Delta X_n$ of β, each X_1 being an element of B_i, let $\omega^{1,\mu}$ be a chain mapping of γ_1 into μ defined as in Lemma 5 which is an extension of the zero-chain map $\omega_{\mu'}$ defined to be $\sum\omega_u(v)$, where the summation extends over all vertices u of μ. That the map $\omega_{\mu'}$ satisfies the hypothesis of Lemma 5 is shown by Lemma 4 and the fact that for the map ω_i as defined in Lemma 3 and for each one-simplex A_i of α_i and maximal coherent subcomplex X_i of β_i each simplex of which has a nonzero coefficient in $\omega_i(A_i)$, the subchain of $\omega_i\delta(A_i)$ over those simplexes of β_i which lie in X_i bounds in X_i. Let ω^1 be the map $\sum\omega^{1,\mu}$, where the summation extends over all n-product simplexes μ of β of the form $X_1\Delta X_2\Delta\cdots\Delta X_n$, each X_i being an element of B_i. ω^1 is a chain mapping of the subcomplex ρ_1 of α into β.

The chain mapping ω will be constructed inductively. Suppose the chain map ω^j of the complex ρ_j into β is defined and has the following properties: ω^j is an extension of ω^{j-1} and for each n-product simplex $\gamma=A_1\Delta A_2\Delta\cdots\Delta A_n$ of ρ_j, if B_i denotes the collection of all maximal coherent subcomplexes of β_i each of whose simplexes has nonzero coefficients in $\omega_i(A_i)$, and for each complex $\mu=Y_1\Delta Y_2\Delta\cdots\Delta Y_n$ of β, each Y_i being an element of B_i, $\omega^j_\mu(\gamma^i)$, for each i-simplex γ^i in γ, denotes the sum of those i-simplexes in μ with the same coefficients as they have in $\omega^j(\gamma^i)$, then ω^j_μ is a chain mapping of γ

into μ which maps bounding zero-cycles into bounding zero-cycles, and $\omega^j(\gamma^i) = \sum \omega_\mu^j(\gamma^i)$, where the summation is over all μ of the form $Y_1 \triangle Y_2 \triangle \cdots \triangle Y_n$, each Y_i being an element of B_i. Then ω^j can be extended to a chain mapping ω^{j+1} of the complex ρ_{j+1} into β in such a way that ω^{j+1} has the same properties for ρ_{j+1} and β as that given above for ω^j with respect to ρ_j and β.

For the n-product simplex $\gamma_{j+1} = A_1 \triangle A_2 \triangle \cdots \triangle A_n$ of ρ_{j+1}, let C_i be defined as was B_i above but for the A_i of γ_{j+1}. For each complex $\lambda = Z_1 \triangle Z_2 \triangle \cdots \triangle Z_n$ in β, each Z_i being an element of C_i, and for each subsimplex γ^k of an n-product proper subsimplex γ of γ_{j+1}, let $\omega_\lambda(\gamma^k) = \sum \omega_\mu^j(\gamma^k)$, where the summation is over all μ of the form $Y_1 \triangle Y_2 \triangle \cdots \triangle Y_n$, each Y_i being an element of B_i and a subcomplex of Z_i. Since each ω_μ^j satisfies the hypothesis of Lemma 5, ω_λ does; since λ is zero-cyclic, ω_λ can be extended to a chain mapping ω_λ^{j+1} of γ_{j+1} into λ. Define ω^{j+1} of γ_{j+1} into β to be $\sum \omega_\lambda^{j+1}$, where the summation is over all λ of the form $Z_1 \triangle Z_2 \triangle \cdots \triangle Z_n$, each Z_i being an element of C_i. For the n-product simplexes of ρ_{j+1} other than γ_{j+1}, let ω^{j+1} be ω^j. Each simplex of ρ_{j+1} which lies in both γ_{j+1} and another n-product simplex in ρ_{j+1} also lies in an n-product simplex of ρ_{j+1} common to both of them. Thus, the image of such a simplex under ω^j is the same as its image under ω^{j+1}. It might be noted that each map ω_λ^{j+1} maps bounding zero-cycles into bounding zero-cycles, and so the chain mapping ω^{j+1} of ρ_{j+1} into β has the same properties for β_{j+1} as has ω^j for ρ_j.

By mathematical induction there exists a chain mapping ω^k of ρ_k into β having similar properties. ρ_k is α. Denote ω^k by ω. Since ω is an extension of ω', it preserves the Kronecker index of zero-cycles. By its construction, ω has property (ii) of the conclusion of this theorem. Since π also preserves Kronecker indices of zero-cycles, $\omega\pi$ does and $\omega\pi \sim 1$.

THEOREM 3. *The Cartesian product of finitely many compact metric chainable continua is a quasi-complex.*

PROOF. Let $M_1, M_2, \cdots, M_n$ denote compact metric chainable continua. For each i, $1 \leqq i \leqq n$, let $U_1^i, U_2^i, \cdots$ be a sequence of chains covering M_i such that U_{j+1}^i is a refinement of U_j^i and U_j^i is a $(1/j)$-chain; let Φ_j^i denote the nerve of U_j^i. For $j<k$, let $\pi_j^{i,k}$ be a simplicial mapping of Φ_k^i into Φ_j^i induced by inclusion and let $\omega_k^{i,j}$ be the chain mapping of Φ_j^i into Φ_k^i as defined in Lemma 3. For each j, let Φ_j be $\Phi_j^1 \triangle \Phi_j^2 \triangle \cdots \triangle \Phi_j^n$ and define mappings π_j^k of Φ_k into Φ_j and ω_k^j of Φ_j into Φ_k as in Lemma 6. Φ_j is the nerve of the covering of $M = M_1 \times M_2 \times \cdots \times M_n$ obtained by taking the Cartesian product

of the coverings $U_j^1, U_j^2, \cdots, U_j^n$, and π_j^k is a projection mapping of Φ_k onto Φ_j. The proof that the continuum M has Property B is just as that in Theorem 2.

The proof that M has Property C is also much like that in Theorem 2. For each integer i, there is an integer j such that if G is a collection of sets in the open covering $U_j = U_j^1 \times U_j^2 \times \cdots \times U_j^n$ having a common point, some element of the covering $U_i = U_i^1 \times U_i^2 \times \cdots \times U_i^n$ contains the union of the elements of the collection G. If U_k is a term of the sequence $U_1, U_2, \cdots$, let U_m be the element of the set $\{U_j\} \cup \{U_k\}$ having the greater subscript. For any simplex σ of Φ_m, there is a simplex ρ of Φ_j containing π_j^m of σ and all simplexes of Φ_m having nonzero coefficients in $\omega_m^j \pi_j^m(\sigma)$. $\pi_i^j(\sigma)$ is a vertex of Φ_i. Hence, Property C is satisfied.

Comments. A conjecture suggested by the theorem that each compact metric continuum which is chainable has the fixed point property is that each compact metric continuum which has arbitrarily small "square-like" coverings ("cube-like," etc.) has the fixed point property. S. Eilenberg pointed out to the author that an example given by Borsuk [2] settles this conjecture in the negative for such continua with arbitrarily small "cube-like" coverings.

Another open question is to determine which of the compact plane continua not separating the plane are not zero-cyclic quasi-complexes. It would also be interesting to know if the Cartesian product of any two quasi-complexes is a quasi-complex.

Added in proof. The proofs of Theorems 1, 2 and 3, with only slight modifications, actually establish somewhat more general results. Let us enlarge the class of chainable continua to include those compact Hausdorff spaces having a cofinal collection ω of finite open coverings such that the elements of each member of ω can be linearly ordered in such a way that two of them intersect if and only if they are adjacent in the ordering. The proofs of Theorems 1, 2 and 3 hold for this class of spaces if instead of using arguments depending on a metric we use the following lemmas (see [6, pp. 19 and 326]):

If X and Y are compact Hausdorff spaces, then every finite open covering of $X \times Y$ has a refinement $\mathfrak{A} \times \mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are finite open coverings of X and Y, respectively, and

If f is a continuous map of the compact Hausdorff space X into itself having no fixed point, there is a finite open covering $\mathfrak{A}$ of X such that no star of $\mathfrak{A}$ meets its image under f.

T. R. Brahana has announced [7] the result that the direct product of two quasi-complexes is a quasi-complex.

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