A FIXED POINT THEOREM

ELDON DYER

Suppose that space is metric. A chain is a finite collection of open sets \( d_1, d_2, \ldots, d_n \) such that \( d_i \) intersects \( d_j \) if and only if \( |i - j| \leq 1 \). If the elements of a chain are of diameter less than a positive number \( \epsilon \), that chain is said to be an \( \epsilon \)-chain. A compact continuum is said to be chainable if for each positive number \( \epsilon \), there is an \( \epsilon \)-chain covering it. R. H. Bing has called \[1\] such continua snake-like. In 1951 O. H. Hamilton showed \[4\] that every compact chainable continuum has the fixed point property; i.e., that if \( f \) is a continuous mapping of such a continuum \( M \) into itself, then some point of \( M \) is its own image under \( f \).

In the present paper it is shown that the Cartesian product of finitely many compact chainable continua has the fixed point property. Since arcs are compact chainable continua, this is a generalization of the Brouwer fixed point theorem. Two other examples of compact chainable continua are the closure of the graph of \( \sin(1/x) \), \( 0 < x \leq 1 \), and the pseudo-arc. Other examples are given in \[1\].

After reading the original manuscript of this paper, A. D. Wallace raised the question as to whether the Cartesian product of finitely many compact chainable continua is a quasi-complex (p. 323 of \[6\]). Rather surprisingly, the answer to this question is in the affirmative. A proof of this theorem is also given here. Since the Cartesian product of finitely many compact chainable continua is zero-cyclic, the fact that such products have the fixed point property is a special case of the Lefschetz fixed point theorem for zero-cyclic quasi-complexes. Since the author’s original argument is of a very different nature, it is also given.

Let \( E^n \) denote Euclidean \( n \)-space and \( \mathbb{R}^n \) the set of all points of \( E^n \) whose distance from the origin is not greater than one. Let \( S^{n-1} \) denote the set of all points of \( E^n \) whose distance from the origin is one. Let \( I \) denote the set of all points on the \( x \)-axis having abscissa \( x \) such that \( 0 \leq x \leq 1 \), and let \( I^n \) denote the set of all points of \( E^n \) each of whose \( n \) coordinates \( x_i \) satisfies \( 0 \leq x_i \leq 1 \). If \( P \) is a point of \( E^n \) having coordinates \( (x_1, x_2, \ldots, x_n) \) and \( t \) is a real number, by \( tP \) is meant the point of \( E^n \) having coordinates \( (tx_1, tx_2, \ldots, tx_n) \).

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For definitions of those concepts from homotopy theory which are used in this paper the reader is referred to §3 of Chapter VI of [5].

**Lemma 1.** Suppose $f$ and $g$ are continuous mappings of $R^n$ into itself and $g|S^{n-1}$ is an essential mapping onto $S^{n-1}$. Then there is a point $x$ in $R^n$ such that $f(x) = g(x)$.

**Proof.** Suppose there is no point $x$ in $R^n$ such that $f(x) = g(x)$. For each point $x$ in $R^n$ let $h(x) = g(x) - f(x)$. For each point $x$ in $S^{n-1}$ and number $t$, $0 \leq t \leq 1$, let $H(x, t) = h(tx)/|h(tx)|$. This is possible since $h(x) \neq 0$ for any $x$ in $R^n$. $H(x, t)$ is continuous in $x$ and $t$.

If $H(x, 1)$ and $g(x)$ were not antipodal for any point $x$ in $S^{n-1}$, they would be homotopic; but $g|S^{n-1}$ is essential and $H(S^{n-1}, 1)$ is homotopic to the constant map $H(S^{n-1}, 0)$. Hence, there is a point $x$ in $S^{n-1}$ such that $H(x, 1) = -g(x)$. Then $g(x) \cdot (1 + |g(x) - f(x)|) = f(x)$. Let $k = 1 + |g(x) - f(x)|$, $k \cdot g(x) = f(x)$. Since $|g(x)| = 1$ and $|f(x)| \leq 1$, $k \leq 1$; but by its definition $k \geq 1$. Therefore, $g(x) = f(x)$.

**Lemma 2.** Suppose that for each integer $i$, $1 \leq i \leq n$, $f_i$ is a continuous mapping of $I$ onto $I$ such that $f_i(0) = 0$ and $f_i(1) = 1$. For each point $x = (x_1, x_2, \cdots, x_n)$ of $I^n$ let $f(x) = (f_1(x_1), f_2(x_2), \cdots, f_n(x_n))$. Let $T^{n-1}$ denote the set of all points $x$ of $I^n$ such that for some $i$, $x_i$ is either $0$ or $1$. Then $f|T^{n-1} = T^{n-1}$, and $f|T^{n-1}$ is homotopic to the identity map of $T^{n-1}$ onto itself.

**Proof.** For each integer $i$, $1 \leq i \leq n$, there is a continuous mapping $h_i(x, t)$ of $I \times I$ onto $I$ such that $h_i(x, 0) = f_i(x)$, $h_i(x, 1) = x$, and $h_i(0, t) = 0$ and $h_i(1, t) = 1$ for $0 \leq t \leq 1$. Let $H(x, t)$, $0 \leq t \leq 1$ and $x$ in $I^n$, be defined as follows: $H(x, t) = (h_1(x_1, 1), \cdots, h_{i-1}(x_{i-1}, 1), h_i(x_i, t), h_{i+1}(x_{i+1}, 0), \cdots, h_n(x_n, 0))$, where $(i-1)/n \leq t \leq i/n$ and $u = n \cdot [t - (i-1)/n]$. It can easily be shown that $H(x, t)$ is continuous in $x$ and $t$, that $H(x, 0) = f(x)$ for all $x$ in $I^n$, that $H(x, 1) = x$ for all $x$ in $T^{n-1}$ (in fact in $I^n$), and that $H(T^{n-1}, t) = T^{n-1}$ for all $t$, $0 \leq t \leq 1$.

**Theorem 1.** Suppose that $M$ is the Cartesian product of $n$ compact chainable continua $X_1, X_2, \cdots, X_n$ and $f$ is a continuous mapping of $M$ into itself. Then there is a point $x$ of $M$ such that $x = f(x)$.

**Proof.** Let $M$ be all of space. Suppose there is no point $x$ of $M$ such that $x = f(x)$. There is a positive number $\varepsilon$ such that $d(x, f(x)) > 4n\varepsilon$ for all points $x$ in $M$. For each integer $i$, $1 \leq i \leq n$, there is a chain $A_i$ covering $X_i$ such that each element of the collection $C_n$ of all ordered $n$-tuples $(a_1, a_2, \cdots, a_n)$, where $a_i$ is in $A_i$, has diameter less than $\varepsilon$. For each integer $i$, $1 \leq i \leq n$, there is a chain $B_i$ covering
Xi such that $B'_i$ is a refinement of $A_i$ and each element of the collection $C'_i$ of all ordered $n$-tuples $(b_1, b_2, \cdots, b_n)$, where $b_i$ is in $B'_i$, is mapped by $f$ into some element of $C_A$. For each $i$ let $B_i$ denote a subchain of $B'_i$ which is irreducible with respect to covering points in the first and last links of $A_i$ which are not in any other link of $A_i$, and let $C_i$ denote the subcollection of $C'_i$ of all ordered $n$-tuples $(b_1, b_2, \cdots, b_n)$, where $b_i$ is in $B_i$. For each $i$ let $a_i$ denote the number of links of $A_i$ and $b_i$ denote the number of links of $B_i$. If for some $i$, the first link of $B_i$ lies in the last link of $A_i$, renumber the links of $B_i$ so that its $j$th link becomes its $(b_i - j + 1)$st link.

For each $x = (x_1, x_2, \cdots, x_n)$, where $x_i = k_i/\beta_i - 1$, $k_i$ being an integer such that $0 \leq k_i \leq \beta_i - 1$, let $\rho_i$ be the element of $C_i$ whose $i$th term is the $(k_i + 1)$st link in $B_i$. $f(\rho_i)$ is in some element $\theta$ of $C_A$. The $i$th term of $\theta$ is the $p_i$th link of $A_i$. There are two adjacent positive integers $m_i$ and $m_i + 1$ such that the $i$th term of any element of $C_A$ containing $f(\rho_i)$ is either the $m_i$th or the $(m_i + 1)$st link of $A_i$, and one of the numbers $m_i$ and $m_i + 1$ is $p_i$. If no element of $C_A$ containing $f(\rho_i)$ has an $i$th term other than the $p_i$th link of $A_i$, let

$$f_i(x) = (p_i - 1)/(a_i - 1).$$

If some element of $C_A$ containing $f(\rho_i)$ has an $i$th term other than the $p_i$th link of $A_i$, let $f_i(x) = (2m_i - 1)/(2(a_i - 1))$. Let

$$F(x) = (f_1(x), f_2(x), \cdots, f_n(x)).$$

Let $B$ denote the set of all points $x = (x_1, x_2, \cdots, x_n)$ of $I^n$ such that for each integer $i$, $1 \leq i \leq n$, there is an integer $k_i$, $0 \leq k_i \leq \beta_i - 1$, such that $x_i = k_i/\beta_i - 1$. Let $\mathcal{B}$ denote the collection of all sets $\sigma$ of $2^n$ points of $B$ for which there is a point $P$ of $\sigma$ such that for every point $Q$ of $\sigma$ and for each integer $i$, $1 \leq i \leq n$, the $i$th coordinate of $Q$ either equals the $i$th coordinate of $P$ or exceeds it by 1. It can be shown (see, for example, §8 of Chapter II of [3]) that there is an $n$-complex $\beta$ which is a triangulation of $I^n$ such that the vertex set of each simplex in $\beta$ is a subset of an element of $\mathcal{B}$.

For each point $x$ in $I^n$ and simplex $\sigma$ of $\beta$ containing it, let $(a_0, a_1, \cdots, a_j)$ be the barycentric coordinates of $x$ with respect to the vertex set $(s_0, s_1, \cdots, s_j)$ of $\sigma$. $x = a_0 s_0 + a_1 s_1 + \cdots + a_j s_j$. Define $F_\sigma(x)$ to be $a_0 F(s_0) + a_1 F(s_1) + \cdots + a_j F(s_j)$. Clearly, $F_\sigma$ is continuous on $\sigma$. If $x$ is common to two simplexes $\sigma_1$ and $\sigma_2$ of $\beta$, let $\sigma$ denote their common face. Then $F_{\sigma_1}(x) = F_\sigma(x) = F_{\sigma_2}(x)$. Hence, if $F(x)$ denotes $F_\sigma(x)$ for any simplex $\sigma$ of $\beta$ containing $x$, $F$ is a continuous mapping of $I^n$ into itself. Furthermore, if $P$ is a vertex of a simplex of $\beta$ containing the point $x$, for each integer $i$, $1 \leq i \leq n$, the
ith coordinates of \( F(P) \) and \( F(x) \) do not differ by more than \( 1/(\alpha_i - 1) \).

Let \( g_i((k-1)/(\beta_i - 1)) = (j-1)/(\alpha_i - 1) \) if the \( k \)th link of \( B_i \) lies in only the \( j \)th link of \( A_i \). If the \( k \)th link of \( B_i \) lies in two links of \( A_i \), the \( j \)th and \((j+1)\)st, let \( g_i((k-1)/(\beta_i - 1)) = (2j-1)/2/(\alpha_i - 1) \). Let \( g_i(I) = I \) be the piecewise linear extension of this mapping. For each point \( x \) of \( I \) such that \( (k-1)/(\beta_i - 1) \leq x \leq k/(\beta_i - 1) \), \( g_i(x) \) is between \( g_i((k-1)/(\beta_i - 1)) \) and \( g_i(k/(\beta_i - 1)) \). For each point \( x \) in \( I^n \), let \( g(x) = (g_1(x_1), g_2(x_2), \ldots, g_n(x_n)) \). Let \( T^{n-1} \) denote the boundary of \( I^n \). By Lemma 2, \( g|T^{n-1} \) is homotopic to the identity mapping of \( T^{n-1} \) onto itself; hence, \( g|T^{n-1} \) is essential onto \( T^{n-1} \).

By Lemma 1, there is a point \( x \) of \( I^n \) such that \( F(x) = g(x) \). Let \( v \) denote a vertex of a simplex of \( \beta \) containing \( x \). For each integer \( i \), \( 1 \leq i \leq n \), the \( i \)th coordinate of \( F(x) \) does not differ from the \( i \)th coordinate of \( F(v) \) by more than \( 1/(\alpha_i - 1) \). Also, the \( i \)th coordinate of \( g(x) \) does not differ from the \( i \)th coordinate of \( g(v) \) by more than \( 1/(\alpha_i - 1) \). Hence, the \( i \)th coordinate of \( F(v) \) does not differ from the \( i \)th coordinate of \( g(v) \) by more than \( 2/(\alpha_i - 1) \). The point \( v \) is an element of the set \( B_i \); i.e., \( v = (v_1, v_2, \ldots, v_n) \), where for each integer \( i \), \( 1 \leq i \leq n \), there is an integer \( k_i, 0 \leq k_i \leq \beta_i - 1 \), such that \( v_i = k_i/(\beta_i - 1) \).

Let \( V \) denote the element of the set \( C_\alpha \) whose \( i \)th term is the \((k_i + 1)\)st link of \( B_i \). Let \( \emptyset \) denote an element of \( C_\alpha \) such that for each \( i \), the \( i \)th term of \( V \) lies in the \( i \)th term of \( \emptyset \) and let \( \emptyset \) be an element of \( C_\alpha \) containing \( f(V) \). For each \( i \) there is a subchain in \( A_i \) from the \( i \)th term of \( \emptyset \) to the \( i \)th term of \( \emptyset \) having not more than \( 4n \) links. Each of these links is of diameter less than \( e \); therefore, \( d(V, f(V)) < 4ne \). This is a contradiction.

**Corollary.** The Cartesian product of the elements of any collection of compact chainable continua has the fixed point property.

**Proof.** This follows immediately from Theorem 1 and the theorem that if \( G \) is a collection of compact Hausdorff spaces such that the Cartesian product of the elements of each finite subcollection of \( G \) has the fixed point property, then the Cartesian product of the elements of the collection \( G \) has the fixed point property.

For the definition of the term quasi-complex the reader is referred to p. 323 of \cite{6}. Notation and terminology used, with only a few exceptions, are in conformity with that used in \cite{6}.

**Lemma 3.** If \( \alpha \) and \( \beta \) are arc-like finite simplicial complexes and \( \pi \) is a simplicial mapping of \( \beta \) onto \( \alpha \), there exists a chain mapping \( \omega \) of \( \alpha \) into \( \beta \) such that

(i) for each zero-chain \( \gamma^0 \) of \( \alpha \), \( KI(\gamma^0) = KI(\omega(\gamma^0)) \);
(ii) for each $i$-simplex $\sigma^i$ of $\alpha$ and $i$-simplex $\gamma^i$ of $\beta$ having nonzero coefficient in $\omega(1 \cdot \sigma^i), \pi(\gamma^i) \subset \sigma^i$; and (iii) $\omega \tau \sim 1$.

Proof. Let $a_1, a_2, \ldots, a_n$ denote the vertices of $\alpha$ ordered as on $\alpha$. There is a subarc $\beta'$ of $\beta$ such that $\pi(\beta') = \alpha$ and there is no proper subarc $\gamma$ of $\beta'$ such that $\pi(\gamma) = \alpha$. Let $b_1$ denote the vertex of $\beta'$ such that $\pi(b_1) = a_1$ and let $b_1, b_2, \ldots, b_m$ denote the vertices of $\beta'$ ordered as on $\beta'$. There is a subsequence $b_{k_1}, b_{k_2}, \ldots, b_{k_p}$ of $b_1, b_2, \ldots, b_m$ such that

1. $\pi(b_{k_j}) = a_1$ and $\pi(b_{k_p}) = a_n$;
2. if $\pi(b_{k_j}) = a_1$ and $\pi(b_{k_{j+1}}) = a_k$, then $|j - k| \leq 1$; and
3. for each $i$, $k_{i+1}$ is the greatest integer $j$ such that
   a. $k_i \leq j \leq k_p$ and
   b. $\pi(b_{k_j}) = a_i$ and $\pi(b_{k_{j+1}}) = a_{i+1}$, then $X_i = -1$.

Define $\omega(1 \cdot a_i)$ to be $1 \cdot \sum_{j=1}^{p} X_i \cdot b_{k_j}$, where $X_i = 0$ if $\pi(b_{k_j}) \neq a_i$,

and if $\pi(b_{k_j}) = a_i$ and $\pi(b_{k_{j+1}}) = a_{i+1}$, then $X_i^j = +1$.

Define $\omega(1 \cdot (a_i, a_{i+1})) = \sum_{j=1}^{m-1} Y_i \cdot (b_j, b_{j+1})$, where if $k_i \leq j < j+1 \leq k_{i+1}$ and for all $l$, $k_i \leq l \leq k_{i+1}$, $\pi(b_l) \in \{a_i\} \cup \{a_{i+1}\}$, then $Y_i^j = +1$; otherwise, $Y_i^j = 0$.

To show that $\omega$ is a chain mapping, it will be shown that for each $j$, the coefficient of $b_j$ is the same in both $\omega(1 \cdot (a_i, a_{i+1}))$ and $\delta\omega(1 \cdot (a_i, a_{i+1}))$, where $\delta$ denotes the boundary operator. Unless $j$ is some $k_i$, its coefficient in both expressions is zero. Suppose $j = k_i$.

Case 1. $\pi(b_{k_i}) = a_{i+1}$. If $\pi(b_{k_{i+1}}) = a_{i+2}$, then $\{\pi(b_{k_{i-1}})\} \cup \{\pi(b_{k_{i+1}})\} \subset \{a_i\} \cup \{a_{i+1}\}$ and $\omega(a_i, a_{i+1})$ contains $1 \cdot (b_{k_{i-1}}, b_{k_i})$ and $0 \cdot (b_{k_i}, b_{k_{i+1}})$. If $\pi(b_{k_{i+1}}) = a_i$, then either $\pi(b_{k_{i-1}})$ or $\pi(b_{k_{i+1}})$ does not lie in $\{a_i\} \cup \{a_{i+1}\}$ and $\omega(a_i, a_{i+1})$ contains $0 \cdot (b_{k_{i-1}}, b_{k_i})$ and $1 \cdot (b_{k_i}, b_{k_{i+1}})$. Thus, $\delta\omega(a_i, a_{i+1})$ contains $+1 \cdot b_{k_i}$ (or $-1 \cdot b_{k_i}$) if $\pi(b_{k_{i+1}}) = a_{i+2}$ (or $a_i$). This is the same coefficient for $b_{k_i}$ as that given by $\omega(1 \cdot a_{i+1})$, which is the same as that given by $\omega\delta(a_i, a_{i+1})$.

Case 2. $\pi(b_{k_i}) = a_i$. If $\pi(b_{k_{i+1}}) = a_{i+1}$, then $\omega(a_i, a_{i+1})$ contains $0 \cdot (b_{k_{i-1}}, b_{k_i})$ and $1 \cdot (b_{k_i}, b_{k_{i+1}})$. If $\pi(b_{k_{i+1}}) = a_{i-1}$, then $\omega(a_i, a_{i+1})$ contains $1 \cdot (b_{k_{i-1}}, b_{k_i})$ and $0 \cdot (b_{k_i}, b_{k_{i+1}})$. Thus, $\delta\omega(a_i, a_{i+1})$ contains $-1 \cdot b_{k_i}$ (or $+1 \cdot b_{k_i}$) if $\pi(b_{k_{i+1}}) = a_{i+1}$ (or $a_{i-1}$). This is the same coefficient for $b_{k_i}$ as that given by $\omega(-1 \cdot a_i)$, which is the same as that given by $\omega\delta(a_i, a_{i+1})$. Thus, $\omega$ is a chain mapping.

To show that $\omega$ preserves the Kronecker index of zero-cycles, it will be shown that $K\!I(\omega(1 \cdot a_i)) = 1$, for each $i$. Clearly, $K\!I(\omega(1 \cdot a_i))$
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Let $U_i$ be the collection of all subarcs $\gamma$ of $\beta'$ which are maximal with respect to $\pi(\gamma) = a_i$ and such that if $b_i, b_{i+1}, \ldots, b_{i+k}$ are the vertices of $\gamma$, $\pi(b_{i-1}) = a_{i-1}$ and $\pi(b_{i+k+1}) = a_{i+1}$. Let $D_i$ be the collection of all such maximal subarcs for which $\pi(b_{i-1}) = a_{i-1}$ and $\pi(b_{i+k+1}) = a_{i+1}$. For each element of $U_i \cup D_i$, the vertex $b_{i+k}$ is a $b_k$, and $\omega(1 \cdot a_i)$ contains $+1 \cdot b_{i+k}$ or $-1 \cdot b_{i+k}$ as $b_{i+k}$ is a vertex of an element of $U_i$ or $D_i$. Also $\omega(1 \cdot a_i)$ does not attach a nonzero coefficient to any $b_k$ which is not a vertex of some element of $U_i \cup D_i$. Thus, $K_1(\omega(1 \cdot a_i))$ equals the number of elements of $U_i$ minus the number of elements of $D_i$. If the elements of $U_i \cup D_i$ are ordered as on $\beta'$, between each two elements of $U_i$ there is an element of $D_i$ and between each two elements of $D_i$ there is an element of $U_i$; furthermore, the first and last elements of $U_i \cup D_i$ are in $U_i$. Hence, $U_i$ has one more element than $D_i$, and $K_1(\omega(1 \cdot a_i)) = 1$.

For cycles $\gamma^p$ on $\beta$ of dimension $p = 1$, $\omega(\gamma^p)$ is a cycle; also $\omega(\gamma^p) - \gamma^p$ is a cycle. Since $\beta$ is zero-cyclic, $\omega(\gamma^p) - \gamma^p \sim 0$ or $\omega(\gamma^p) \sim \gamma^p$. For any zero-cycle, $\gamma^0$, on $\beta$, $K_1(\omega(\gamma^0)) = K_1(\gamma^0)$, and so $\omega(\gamma^0) \sim \gamma^0$. Hence, $\gamma^0 \sim 1$.

**Theorem 2.** Every compact metric chainable continuum is a quasi-complex.

**Proof.** Let $M$ denote a compact metric chainable continuum. Let $U_1, U_2, \ldots$ be a sequence of chains covering $M$ such that $U_{i+1}$ is a refinement of $U_i$ and $U_i$ is a $(1/i)$-chain; let $\Phi_i$ denote the nerve of $U_i$. If $i$ and $j$ are positive integers and $i < j$, let $\pi_i^j$ denote a simplicial mapping of $\Phi_j$ onto $\Phi_i$ induced by inclusion; i.e., a projection mapping. Let $\omega^j_i$ denote the chain mapping of $\Phi_i$ into $\Phi_j$ defined for $\pi_i^j$ as in Lemma 3. Antiprojections will be the mappings $\omega^j_i$ and finite products $\omega_i^{m-1} \cdots \omega_i^1 \cdot \omega_i^0$, where $i < i_1 < \cdots < i_n$. These products also preserve the Kronecker index of zero-cycles on $\Phi_i$. These mappings have all of the properties required in Property B on p. 323 of [6]. To show that $M$ has Property C, for each $U_i(=a)$, let $U_j(=g)$ be a sufficiently small refinement of $U_i$ that any three adjacent elements of $U_j$ lie in some element of $U_i$. Then for any $U_k(=b)$, let $U_m(=h)$ be the one of the two $U_j$ and $U_k$ which is a refinement of both. Therefore, $M$ has Property C.

For two complexes $K_1$ and $K_2$, $K_1 \Delta K_2$ denotes the simplicial product of $K_1$ and $K_2$ as defined in §8 of Chapter II of [3].

**Lemma 4.** If for each $i$, $1 \leq i \leq n$, $\alpha_i$ is a finite simplicial complex, $\beta_i$ is a connected finite simplicial complex, and $\omega_i$ maps the zero-chains of $\alpha_i$ into zero-chains of $\beta_i$ in such a way that $K_1(\omega_i(\sigma^i)) = 0$ for each one-simplex $\sigma^i$ of $\alpha_i$, then if $\alpha = \alpha_1 \Delta \alpha_2 \Delta \cdots \Delta \alpha_n$, $\beta = \beta_1 \Delta \beta_2 \Delta \cdots \Delta \beta_n$. 
\[ \Delta \beta_n, \text{ and for each vertex } v = (a_1, a_2, \ldots, a_n) \text{ of } \alpha, \text{ where } a_i \text{ is a vertex of } \alpha, \quad \omega(1 \cdot v) = \sum \gamma_b \cdot (b_1, b_2, \ldots, b_n), \text{ where the summation extends over all vertices } b = (b_1, b_2, \ldots, b_n) \text{ of } \beta, \text{ and for each } b, \gamma_b \text{ is the product of the coefficients of the } b_i \text{ in } \omega(a_i), \text{ then if } \gamma^0 \text{ is a bounding zero-cycle in } \alpha, \omega(\gamma^0) \text{ is a bounding zero-cycle in } \beta. \]

**Proof.** For each vertex \( v \) of \( \alpha \), the Kronecker index of \( \omega(v) \) equals the product of the Kronecker indices of the \( \omega_i(a_i) \), where \( \{a_i\} \) are the coordinates of \( v \). If \( \gamma^0 \) is the boundary of the one-simplex \((v_1, v_2)\) of \( \alpha \), where all of the coordinates of \( v_1 \) and \( v_2 \) are the same except the \( i \)th, then \( \text{KI}(\omega(\gamma^0)) = \text{KI}(\omega(v_2)) - \text{KI}(\omega(v_1)) = 0 \), since \( \text{KI}(\omega(\delta(v_1, v_2))) = 0 \). Since \( \beta \) is connected, \( \omega(\gamma^0) \) bounds. If \( \gamma^0 \) is the boundary of any one-simplex in \( \alpha \), it is the sum of the boundaries of one-simplexes in \( \alpha \) of the sort discussed in the previous sentence; hence, \( \text{KI}(\omega(\gamma^0)) = 0 \). Finally, if \( \gamma^0 = \delta \sum_{i=1}^n p_k \cdot \sigma_i^k \), where each \( \sigma_i^k \) is a one-simplex of \( \alpha \) and \( p_k \) is an element of the coefficient group, \( \omega(\gamma^0) = \omega \left( \sum_{i=1}^n p_k \cdot \delta(\sigma_i^k) \right) = \sum_{i=1}^n p_k \cdot \omega(\sigma_i^k) \); therefore, \( \text{KI}(\omega(\gamma^0)) = 0 \). Hence, in each case \( \omega(\gamma^0) \) bounds if \( \gamma^0 \) bounds.

**Lemma 5.** If \( \sigma \) is a complex, \( \sigma' \) is a subcomplex of \( \sigma \) containing all of the vertices of \( \sigma \), \( \gamma \) is a zero-cyclic complex, and \( \omega' \) is a chain mapping of \( \sigma' \) into \( \gamma \) which maps bounding zero-cycles on \( \sigma' \) into bounding zero-cycles on \( \gamma \), then \( \omega' \) can be extended to a chain mapping \( \omega \) of \( \sigma \) into \( \gamma \).

**Proof.** Let \( \alpha_1, \alpha_2, \ldots, \alpha_k \) denote the simplexes in \( \sigma \) which are not in \( \sigma' \) ordered such that if \( i < j \), then the dimension of \( \alpha_i \) is less than or equal to the dimension of \( \alpha_j \). The argument will be by mathematical induction. For each \( i, 1 \leq i \leq k \), let \( \sigma_i \) denote the subcomplex of \( \sigma \) consisting of all simplexes of \( \sigma' \) and all simplexes in each of the simplexes \( \alpha_1, \alpha_2, \ldots, \alpha_i \). \( \omega_1 \) is the extension of \( \omega' \) to \( \sigma_1 \) obtained as follows: since \( \delta(\alpha_1) \) is in \( \sigma' \) and is a bounding cycle, \( \omega' \delta(\alpha_1) \) is a cycle in \( \gamma \) which bounds in \( \gamma \) if it is a zero-cycle since it is the image of a bounding zero-cycle and bounds in \( \gamma \) if it is a higher-dimensional cycle since all such cycles bound in \( \gamma \). Let \( \tau_1 \) denote a chain in \( \gamma \) such that \( \omega' \delta(\alpha_1) = \delta \tau_1 \) and let \( \omega_1(\alpha_1) = \tau_1 \). Then \( \omega_1 \) is a chain mapping of \( \sigma_1 \) into \( \gamma \) and is an extension of \( \omega' \). It is clear this process can be continued inductively.

An \( n \)-product simplex is a simplex which is the \( \Delta \) product of \( n \) simplexes.

**Lemma 6.** If for each \( i, 1 \leq i \leq n \), \( \alpha_i \) and \( \beta_i \) are arc-like finite simplicial complexes and \( \pi_i \) is a simplicial mapping of \( \beta_i \) onto \( \alpha_i \), then if for each vertex \( b \) of \( \beta = \beta_1 \Delta \beta_2 \Delta \cdots \Delta \beta_n, \pi(b) \) is the vertex \((\pi_1(b_1), \pi_2(b_2), \ldots, \pi_n(b_n)) \) of \( \alpha = \alpha_1 \Delta \alpha_2 \Delta \cdots \Delta \alpha_n \), where for each \( i, b_i \) is the \( i \)th coordinate.
of \( b \), then \( \pi \) is a simplicial mapping of \( \beta \) onto \( \alpha \) and there is a chain mapping \( \omega \) of \( \alpha \) into \( \beta \) such that

(i) for each zero-chain \( \gamma^0 \) of \( \alpha \), \( \text{KI}(\gamma^0) = \text{KI}(\omega(\gamma^0)) \);

(ii) for each \( k \)-simplex \( \sigma^k \) of \( \alpha \), \( (2^n - 1) \)-simplex \( \sigma \) of \( \alpha \) of which \( \sigma^k \) is a face, and \( k \)-simplex \( \gamma^k \) of \( \beta \) having nonzero coefficient in \( \omega(1 \cdot \sigma^k) \), \( \pi(\gamma^k) \subseteq \sigma \); and

(iii) \( \omega \pi \sim 1 \).

Proof. For each vertex \( v = (a_1, a_2, \cdots, a_n) \) of \( \alpha \) and vertex \( u = (b_1, b_2, \cdots, b_n) \) of \( \beta \), let \( \omega_u(v) = \lambda_u \cdot u \), where \( \lambda_u \) is the product of the coefficients of the \( b_i \) in \( \omega_i(a_i) \), \( \omega_i \) being defined for \( \alpha_i, \beta_i, \) and \( \pi_i \) as in Lemma 3. Let \( \omega'(v) = \sum \omega_u(v) \), where the summation is over all vertices \( u \) in \( \beta \). \( \text{KI}(\omega'(v)) \) is the product of the Kronecker indices of the \( \omega_i(a_i) \) and since each of these is one, \( \text{KI}(\omega'(v)) = 1 \).

Let \( \gamma_1, \gamma_2, \cdots, \gamma_k \) denote the \( n \)-product simplexes of \( \alpha \) of positive dimension ordered so that if \( i < j \), then the dimension of \( \gamma_i \) is less than or equal to the dimension of \( \gamma_j \). For each \( j \), let \( \rho_j \) denote the subcomplex of \( \alpha \) composed of all vertices of \( \alpha \) and all faces of all simplexes \( \gamma_1, \gamma_2, \cdots, \gamma_j \). \( \gamma_i = A_1 \Delta A_2 \Delta \cdots \Delta A_n \), where each \( A_i \) is a simplex in \( \alpha_i \). Let \( B_i \) denote the collection of all maximal coherent subcomplexes of \( \beta_i \) each simplex of which has a nonzero coefficient in \( \omega_i(a_i) \). For each subcomplex \( \mu = X_1 \Delta X_2 \Delta \cdots \Delta X_n \) of \( \beta_i \), each \( X_1 \) being an element of \( B_i \), let \( \omega^i,\mu \) be a chain mapping of \( \gamma_i \) into \( \mu \) defined as in Lemma 5 which is an extension of the zero-chain map \( \omega^i \) defined to be \( \sum \omega_u(v) \), where the summation extends over all vertices \( u \) of \( \mu \). That the map \( \omega^i,\mu \) satisfies the hypothesis of Lemma 5 is shown by Lemma 4 and the fact that for the map \( \omega_i \) as defined in Lemma 3 and for each one-simplex \( A_i \) of \( \alpha_i \) and maximal coherent subcomplex \( X_i \) of \( \beta_i \) each simplex of which has a nonzero coefficient in \( \omega_i(A_i) \), the subchain of \( \omega_i(\delta(A_i)) \) over those simplexes of \( \beta_i \) which lie in \( X_i \) bounds in \( X_i \). Let \( \omega^i \) be the map \( \sum \omega^i,\mu \), where the summation extends over all \( n \)-product simplexes \( \mu \) of \( \beta \) of the form \( X_1 \Delta X_2 \Delta \cdots \Delta X_n \), each \( X_i \) being an element of \( B_i \). \( \omega^i \) is a chain mapping of the subcomplex \( \rho_i \) of \( \alpha \) into \( \beta \).

The chain mapping \( \omega \) will be constructed inductively. Suppose the chain map \( \omega^i \) of the complex \( \rho_i \) into \( \beta \) is defined and has the following properties: \( \omega^i \) is an extension of \( \omega^{i-1} \) and for each \( n \)-product simplex \( \gamma = A_1 \Delta A_2 \Delta \cdots \Delta A_n \) of \( \rho_i \) if \( B_i \) denotes the collection of all maximal coherent subcomplexes of \( \beta_i \) each of whose simplexes has nonzero coefficients in \( \omega_i(A_i) \), and for each complex \( \mu = Y_1 \Delta Y_2 \Delta \cdots \Delta Y_n \) of \( \beta \), each \( Y_i \) being an element of \( B_i \), \( \omega^i,\mu(\gamma^i) \), for each \( i \)-simplex \( \gamma_i \) in \( \gamma \), denotes the sum of those \( i \)-simplexes in \( \mu \) with the same coefficients as they have in \( \omega^i(\gamma^i) \), then \( \omega^i,\mu(\gamma^i) \) is a chain mapping of \( \gamma_i \).
into $\mu$ which maps bounding zero-cycles into bounding zero-cycles, and $\omega^i(\gamma^i) = \sum \omega^i_\mu(\gamma^i)$, where the summation is over all $\mu$ of the form $Y_1 \Delta Y_2 \Delta \cdots \Delta Y_n$, each $Y_i$ being an element of $B_i$. Then $\omega^i$ can be extended to a chain mapping $\omega^{i+1}$ of the complex $\rho_{j+1}$ into $\beta$ in such a way that $\omega^{i+1}$ has the same properties for $\rho_{j+1}$ and $\beta$ as that given above for $\omega^i$ with respect to $\rho_j$ and $\beta$.

For the $n$-product simplex $\gamma_{j+1} = A_1 \Delta A_2 \Delta \cdots \Delta A_n$ of $\rho_{j+1}$, let $C_i$ be defined as was $B_i$ above but for the $A_i$ of $\gamma_{j+1}$. For each complex $\lambda = Z_1 \Delta Z_2 \Delta \cdots \Delta Z_n$ in $\beta$, each $Z_i$ being an element of $C_i$, and for each subsimplex $\gamma^h$ of an $n$-product proper subsimplex $\gamma$ of $\gamma_{j+1}$, let $\omega_\lambda(\gamma^h) = \sum \omega_{\lambda}(\gamma^h)$, where the summation is over all $\mu$ of the form $Y_1 \Delta Y_2 \Delta \cdots \Delta Y_n$, each $Y_i$ being an element of $B_i$ and a sub-complex of $Z_i$. Since each $\omega^i_\mu$ satisfies the hypothesis of Lemma 5, $\omega_\lambda$ does; since $\lambda$ is zero-cyclic, $\omega_\lambda$ can be extended to a chain mapping $\omega^{i+1}_\lambda$ of $\gamma_{j+1}$ into $\lambda$. Define $\omega^{i+1}_j$ of $\gamma_{j+1}$ into $\beta$ to be $\sum \omega^{i+1}_\lambda$, where the summation is over all $\lambda$ of the form $Z_1 \Delta Z_2 \Delta \cdots \Delta Z_n$, each $Z_i$ being an element of $C_i$. For the $n$-product simplexes of $\rho_{j+1}$ other than $\gamma_{j+1}$, let $\omega^{i+1}$ be $\omega^i$. Each simplex of $\rho_{j+1}$ which lies in both $\gamma_{j+1}$ and another $n$-product simplex in $\rho_{j+1}$ also lies in an $n$-product simplex of $\rho_{j+1}$ common to both of them. Thus, the image of such a simplex under $\omega^i$ is the same as its image under $\omega^{i+1}$. It might be noted that each map $\omega^{i+1}_\lambda$ maps bounding zero-cycles into bounding zero-cycles, and so the chain mapping $\omega^{i+1}$ of $\rho_{j+1}$ into $\beta$ has the same properties for $\beta_{j+1}$ as has $\omega^i$ for $\rho_j$.

By mathematical induction there exists a chain mapping $\omega^\alpha$ of $\rho_k$ into $\beta$ having similar properties. $\rho_1$ is $\alpha$. Denote $\omega^\alpha$ by $\omega$. Since $\omega$ is an extension of $\omega^i$, it preserves the Kronecker index of zero-cycles. By its construction, $\omega$ has property (ii) of the conclusion of this theorem. Since $\pi$ also preserves Kronecker indices of zero-cycles, $\omega \pi$ does and $\omega \pi \sim 1$.

**Theorem 3.** The Cartesian product of finitely many compact metric chainable continua is a quasi-complex.

**Proof.** Let $M_1, M_2, \ldots, M_n$ denote compact metric chainable continua. For each $i$, $1 \leq i \leq n$, let $U^i_1, U^i_2, \ldots$ be a sequence of chains covering $M_i$ such that $U^i_{j+1}$ is a refinement of $U^i_j$ and $U^i_j$ is a $(1/j)$-chain; let $\Phi_j^i$ denote the nerve of $U^i_j$. For $j < k$, let $\pi^i_{jk}$ be a simplicial mapping of $\Phi_j^i$ into $\Phi_k^i$ induced by inclusion and let $\omega^i_k$ be the chain mapping of $\Phi_k^i$ into $\Phi_1^i$ as defined in Lemma 3. For each $j$, let $\Phi_j$ be $\Phi^1_j \Delta \Phi^2_j \Delta \cdots \Delta \Phi^n_j$ and define mappings $\pi^i_j$ of $\Phi_k$ into $\Phi_j$ and $\omega^i_j$ of $\Phi_j$ into $\Phi_1$ as in Lemma 6. $\Phi_j$ is the nerve of the covering of $M = M_1 \times M_2 \times \cdots \times M_n$ obtained by taking the Cartesian product.
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of the coverings \( U_1, U_2, \ldots, U^n \), and \( \pi_j^i \) is a projection mapping of \( \Phi \) onto \( \Phi_j \). The proof that the continuum \( M \) has Property B is just as that in Theorem 2.

The proof that \( M \) has Property C is also much like that in Theorem 2. For each integer \( i \), there is an integer \( j \) such that if \( G \) is a collection of sets in the open covering \( U_i = U^i_1 \times U^i_2 \times \cdots \times U^i_n \) having a common point, some element of the covering \( U_i = U^i_1 \times U^i_2 \times \cdots \times U^i_n \) contains the union of the elements of the collection \( G \). If \( U_k \) is a term of the sequence \( U_1, U_2, \ldots \), let \( U_m \) be the element of the set \( \{ U_j \} \cup \{ U_k \} \) having the greater subscript. For any simplex \( \sigma \) of \( \Phi_m \), there is a simplex \( \rho \) of \( \Phi_j \) containing \( \pi_m^j(\sigma) \) and all simplexes of \( \Phi_m \) having nonzero coefficients in \( \omega_m^j(\sigma) \). \( \pi_j(\sigma) \) is a vertex of \( \Phi_j \). Hence, Property C is satisfied.

Comments. A conjecture suggested by the theorem that each compact metric continuum which is chainable has the fixed point property is that each compact metric continuum which has arbitrarily small "square-like" coverings ("cube-like," etc.) has the fixed point property. S. Eilenberg pointed out to the author that an example given by Borsuk [2] settles this conjecture in the negative for such continua with arbitrarily small "cube-like" coverings.

Another open question is to determine which of the compact plane continua not separating the plane are not zero-cyclic quasi-complexes. It would also be interesting to know if the Cartesian product of any two quasi-complexes is a quasi-complex.

*Added in proof.* The proofs of Theorems 1, 2 and 3, with only slight modifications, actually establish somewhat more general results. Let us enlarge the class of chainable continua to include those compact Hausdorff spaces having a cofinal collection \( \omega \) of finite open coverings such that the elements of each member of \( \omega \) can be linearly ordered in such a way that two of them intersect if and only if they are adjacent in the ordering. The proofs of Theorems 1, 2 and 3 hold for this class of spaces if instead of using arguments depending on a metric we use the following lemmas (see [6, pp. 19 and 326]):

*If \( X \) and \( Y \) are compact Hausdorff spaces, then every finite open covering of \( X \times Y \) has a refinement \( A \times B \), where \( A \) and \( B \) are finite open coverings of \( X \) and \( Y \), respectively, and*

*If \( f \) is a continuous map of the compact Hausdorff space \( X \) into itself having no fixed point, there is a finite open covering \( A \) of \( X \) such that no star of \( A \) meets its image under \( f \).*

T. R. Brahana has announced [7] the result that the direct product of two quasi-complexes is a quasi-complex.
Bibliography


University of Georgia