ON THE CHARACTERISTIC POLYNOMIAL OF THE
PRODUCT OF SEVERAL MATRICES

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We shall prove two theorems.

THEOREM I. If A is an $n \times n$ matrix with elements in the field $F$, if
$R$ and $S_i$, $i=1, 2, \ldots, r$, are $1 \times n$ matrices with elements in $F$, and
$D_i=R^T S_i$, where $R^T$ is the transpose of $R$, and if the characteristic poly-
nomial of $A_i=A+D_i$ is

$$|x I - A_i| = m_{i0} + m_{i1}x + m_{i2}x^2 + \cdots + m_{i,r-1}x^{r-1},$$

where $m_{i,i-1}$, $i, j=1, 2, \ldots, r$, are polynomials in $x^r$ with coefficients
in $F$, then the characteristic polynomial of the product $P=A_1A_2 \cdots A_r$
is given by $(-1)^{(r-1)} \Delta(x)$, where

$$\Delta(x) = \begin{bmatrix}
m_{10} & m_{11}x & m_{12}x^2 & \cdots & m_{1,r-1}x^{r-1} \\
m_{21}x & m_{20} & m_{21}x & \cdots & m_{2,r-1}x^{r-2} \\
m_{32}x^2 & m_{31}x & m_{30} & \cdots & m_{3,r-1}x^{r-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m_{r,r-1}x^{r-1} & m_{r,r-2}x^{r-2} & m_{r,r-3}x^{r-3} & \cdots & m_{r,0}
\end{bmatrix}.$$

This proposition has been proved by the writer [1] for the case
gave an alternate proof of it and extended his method to the product
of three matrices. This latter result does not come under the theorem
above. Schneider [5] proved the theorem for the case $A_iA_j=0$, $i<j$, $i, j=1, 2, \ldots, r$.

Capital letters and expressions in bold faced parentheses will indicate
matrices with elements in the field $F$ or in $F(\omega)$, the extension of
$F$ by the adjunction of a primitive $r$th root of unity to it, and in
$F(x)$ the polynomial domain of $F(\omega)$. The direct product of $B$ and $C$
is $(b_{ij}C) = B(C)$. The product indicated by $\Pi$ will run from 1 to $r$.

If $R$ is not zero a nonsingular matrix $Q$ with elements in $F$ exists
such that $QR^T = (1, 0, \ldots, 0)^T$; as a result

$$QD_iQ^T = (1, 0, \ldots, 0)^T S_i Q = E_i,$$

where $Q^T$ is the inverse of $Q$ and $E_i$ has nonzero elements in only the
first row. Now let

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(2) \[ QA_kQ^T = M_k = Q(A + D_k)Q^T = M + E_k, \]
where \( Q \) is the matrix defined above and \( QAQ^T = M = (m_{ij}) \). Consequently \( M_k = (m_{ij}^{(k)}) \), where \( m_{ij}^{(k)} = m_{ij} + e_{ij}^{(k)} \) and \( m_{ij}^{(0)} = m_{ij} \) for \( i > 1 \). That is, the matrices \( M_k \) differ only in the elements of their first rows. As a result the elements of the first columns of the adjoints \([xI - M_k]^A \) and \([xI - M]^A \) of \( xI - M_k \) and \( xI - M \) respectively are identical for \( k = 1, 2, \cdots, r \), since all these matrices agree in the elements of their last \( n - 1 \) rows and for the same reason

\[
N_k(x) = \begin{bmatrix} xI - M_k \end{bmatrix} \begin{bmatrix} xI - M \end{bmatrix}^A,
\]

\[
= \begin{bmatrix} m_k(x) & * & * & \cdots & * \\ 0 & m(x) & 0 & \cdots & 0 \\ 0 & 0 & m(x) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & m(x) \end{bmatrix},
\]

\( k = 1, 2, \cdots, r \), where asterisks indicate nonzero elements in \( F(x) \) and \( |xI - M| = m(x) \).

Let

\[
W = (\omega_{ij}) = (\omega^{(i-1)(j-n)}), \quad i, j = 1, 2, \cdots, r;
\]

then

\[
|W(I_k)| = |W|^k,
\]

where \( I_k \) is the identity matrix of order \( k \). The determinantal equation holds because \( W(I_k) \) can be transformed by the interchange of rows and corresponding columns to the direct sum \( W + W + \cdots + W \) of \( k \) summands.

We shall operate in \( F(x) \) on the matrix

\[
M(x) = (\delta_{ij}xI - \delta_{i+1,j}M_i) \quad (\delta_{r+1,1} = 1)
\]

\[
= \begin{bmatrix} xI - M_1 & 0 & \cdots & 0 \\ 0 & xI - M_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots - M_{r-1} \\ - M_r & 0 & 0 & \cdots & xI \end{bmatrix}
\]

If we multiply this matrix on the right by one whose first row is \( I \), \( M_1x^{-1}, M_1M_2x^{-2}, \cdots, M_1M_2 \cdots M_{r-1}x^{-r+1} \), whose second row is \( 0, I, M_2x^{-1}, \cdots, M_2M_3 \cdots M_{r-1}x^{-r+2} \), and whose last row is
0, 0, 0, \ldots, I, \) we find that the determinant of the product is 
\[ | x^r - M_i M_{i+1} \cdots M_{n-1} | \] and is therefore equal to \( | x^r - P | \). The proof of Theorem I will consist in showing that

\[ | M(x) | = (-1)^{(r-1)n} | \Delta(x^r) | . \]

We now proceed to establish this equation.

\[ M(x) W(I) = (\delta_{ij} x I - \delta_{i+1,k} M_i)(\omega(k-1)(1-\rho) I), \]

\[ = (\omega^{i-1}(1-\rho) x I - \omega^{i(1-\rho)} M_i), \]

\[ = (\omega^{i-1}\{ \omega_{ij} [\omega^{i-1} x I - M_i] \}). \]

The number \( \omega^{i-j} \) is a common multiplier of the \( n \times n \) matrices in the \( j \)th column of the \( nr \times nr \) matrix in right member above. Consequently the determinant of this matrix has the factor \( r^r \omega^{(1-k)n} = \omega^{r(r-1)n/2} = \omega^{r(r-1)n/2} = (-1)^{(r-1)n} \). The determinantal equation obtained from the matric equation above is as a result:

\[ | M(x) W | = (-1)^{(r-1)n} | \omega_{ij} [\omega^{i-1} x I - M_i] | . \]

According to (3) the product

\[ (\omega_{ik} [\omega^{k-1} x I - M_i])(\delta_{ij} [\omega^{i-1} x I - M]^{4}) = (\omega_{ij} V_i (\omega^{i-1} x)). \]

The \( nr \times nr \) matrix of the right member of this equation can be transformed by the interchange of corresponding rows and columns to a similar one having the form

\[
\begin{bmatrix}
(\omega_{ij} m_i(\omega^{i-1} x)), & \cdots, & * \\
0, & (\omega_{ij} m_i(\omega^{i-1} x)), & \cdots, & 0 \\
0, & \cdots, & \cdots, & 0 \\
0, & \cdots, & \cdots, (\omega_{ij} m_i(\omega^{i-1} x))
\end{bmatrix},
\]

where asterisks represent \( r \times r \) matrices with elements in \( F(x) \) and the zeros are \( r \times r \) zero matrices. The determinant of this matrix is

\[ | W | = | \prod m_{i}(\omega^{i-1} x) | (\omega_{ij} m_i(\omega^{i-1} x)) | \]

for each of the matrices \( (\omega_{ij} m_i(\omega^{i-1} x)) \) has \( m(\omega^{i-1} x) \) as a divisor of all elements in the \( j \)th column. The determinant of the direct sum \( (\delta_{ij} [\omega^{i-1} x I - M]^{4}) \) in equation (7) is \( | \prod m_{i}(\omega^{i-1} x) |^{n-1}; \) consequently the determinantal equation which follows from (7) and (8) is

\[ | \omega_{ij} [\omega^{i-1} x I - M_i] | [ \prod m_{i}(\omega^{i-1} x) ]^{n-1} \]

\[ = | W |^{n-1} [ \prod m_{i}(\omega^{i-1} x) ]^{n-1} | (\omega_{ij} m_i(\omega^{i-1} x)) | , \]

or

\[ | \omega_{ij} [\omega^{i-1} x I - M_i] | = | W |^{n-1} | (\omega_{ij} m_i(\omega^{i-1} x)) | , \]

\[ (9) \]

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where the determinant of the left member is that of an $nr \times nr$ matrix and those in the right members are of order $r$. From (6) and (9) we have $|M(x)| = (-1)^{(r-1)n} |(\omega_i M_i(\omega^{-1} x))| / |W|$. It remains to be shown that the right member here is $(-1)^{(r-1)n} |\Delta(x')| ;$ this is easily accomplished by multiplying $\Delta(x')$ in (1) on the right by $W$. Herewith equation (5) is established and the proof of Theorem I is completed.

**Corollary.** Under the hypotheses of Theorem I and if $B$ is an $n \times n$ matrix with elements in $F$ and if $B_i = B + S_i^T R$ and
\[
|xI - B_i| = n_{i0} + n_{i1}x + n_{i2}x^2 + \cdots + n_{i,r-1}x^{r-1};
\]
then the characteristic polynomial of $B_1 B_2 \cdots B_r$ is given by
\[
(-1)^{(r-1)n} |\Delta'(x)| \quad \text{where}
\]
\[
\Delta'(x') = \begin{vmatrix}
 n_{r,0}, & n_{r,r-1}x^{-1}, & \cdots, & n_{r,1}x \\
 n_{r-1,1}, & n_{r-1,0}, & \cdots, & n_{r-1,2}x^2 \\
 \cdots & \cdots & \cdots & \cdots \\
 n_{1,r-3}x^{-1}, & n_{1,r-2}x^{-2}, & \cdots, & n_{1,0} \\
\end{vmatrix}.
\]

This case can be made to come under Theorem I for $B_i^T = B_i^T + R^T S_i$, where $B_i^T$ now satisfies the conditions imposed upon $A_i$. Moreover $|xI - B_i| = |xI - B_i^T|$. Since $(B, B_2 \cdots B_r)^T = B_i^T B_i^T \cdots B_i^T$, it follows that in $\Delta(x')$ of (1) we must replace the elements $m_{i,j-1}x^{-1}$ by $n_{r-i+1, j-1}x^{-1}$ in forming the matrix $\Delta'(x')$ above. This proves the corollary.

**Theorem II.** If $D_i$, $i = 1, 2, \cdots, r$, are $n \times n$ matrices with elements in $F$, each of which is nilpotent and commutative with the others and with $A$, which also has elements in $F$, then the characteristic polynomials of $A_i = A + D_i$, $i = 1, 2, \cdots, r$, are given by
\[
|xI - A| = m_0 + m_1x + m_2x^2 + \cdots + m_rx^{r-1},
\]
where $m_{i-1}, i = 1, 2, \cdots, r$, are polynomials in $x'$ with coefficients in $F$, and the characteristic polynomial of the product $P = A_1 A_2 \cdots A_r$ is $(-1)^{(r-1)n} |\Delta(x)|$, where
\[
\Delta(x) = \begin{vmatrix}
 m_0, & m_{r-1}x^{-1}, & m_{r-2}x^{-2}, & \cdots, & m_1x \\
 m_1x, & m_0, & m_{r-1}x^{-1}, & \cdots, & m_2x^2 \\
m_2x^2, & m_1x, & m_0, & \cdots, & m_3x^3 \\
 \cdots & \cdots & \cdots & \cdots & \cdots \\
m_{r-1}x^{-1}, & m_{r-2}x^{-2}, & m_{r-3}x^{-3}, & \cdots, & m_0 \\
\end{vmatrix}.
\]

According to a theorem by Frobenius [4], the determinant of the
matrix $B + C$ is equal to that of $B$ if $B$ and $C$ are commutative matrices and $C$ is nilpotent. This establishes equation (10) as giving the characteristic polynomial of $A_i, i = 1, 2, \ldots, r$. By Theorem I the determinant
\[ |xI - A^r| = (-1)^{(r-1)n} |\Delta(x)|. \]
We shall proceed by induction. Let $P_i = A_1A_2\cdots A_i$, then
\[ |xI - P_iA^{r-i-1}| = |xI - (A + D)A^{r-i-1}| = |xI - A^r - D_iA^{r-i-1}|. \]
Now the matrix $D_iA^{r-1}$ is nilpotent and commutative with $xI - A^r$ consequently the determinants above are equal to $|xI - A^r|$. We assume that
\[ |xI - P_iA^{r-i}| = |xI - A^r|; \]
then
\[ |xI - P_{i+1}A^{r-i-1}| = |xI - P_iA^{r-i} - P_iD_{i+1}A^{r-i-1}|. \]
Here $P_iD_{i+1}A^{r-i-1}$ is commutative with $xI - P_iA^{r-i}$ and is nilpotent because $D_{i+1}$ is nilpotent and commutative with both $P_i$ and $A$; hence by Frobenius' theorem
\[ |xI - P_{i+1}A^{r-i-1}| = |xI - P_iA^{r-i}| = |xI - A^r|. \]
Consequently
\[ |xI - P| = |xI - A^r| = (-1)^{(r-1)n} |\Delta(x)|, \]
and the theorem is proved.

References


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