Without significant loss of generality we may take \( \mu_1 = 1 \). Determining \( g(u), A(y), \psi(t) \) from the differential equations (14)(i), (ii) and the recurrence relation (12)(i) leads to known generating functions.

For \( \alpha - \beta = 0 \) we have the ultraspherical polynomials with known generating functions of Appell type.

**Reference**


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**APPROXIMATION BY FAMILIES OF FUNCTIONS**

**LEONARD TORNHEIM**

Let \( F \) be a set of real functions on the interval \( a \leq x \leq b \) and let \( g(x) \) be a real and continuous function on the same interval. The problem of finding a function in \( F \) which approximates \( g \) best according to some criterion has been extensively studied, especially in the two cases when \( F \) is the set of all real polynomials of degree less than \( n \) or a fixed set of trigonometric functions [1; 2; 4, pp. 40–41]. We investigate the problem for \( n \)-parameter families of functions [5] with reference to some of the usual approximation criteria.

An \( n \)-parameter family \( F \) of functions on the interval \( a \leq x \leq b \) is a set of real continuous functions \( f \) such that for every set of points \( (x_i, y_i) (i = 1, \ldots, n) \) with \( a \leq x_1 < \cdots < x_n \leq b \) there exists exactly one \( f \) in \( F \) with \( f(x_i) = y_i (i = 1, \ldots, n) \).

An \( n \)-parameter family is linear if it is a real vector space, i.e., if there exist \( n \) functions \( f_1, \ldots, f_n \) such that every \( f \) may be expressed as a linear combination \( f = a_1 f_1 + \cdots + a_n f_n \), where \( a_1, \ldots, a_n \) are real.

For \( k \geq 1 \) the modulus of approximation \( M^{(k)} \) of \( g \) by \( f \) is defined as

\[
M^{(k)}(f) = \int_a^b |f - g|^k \, dx \quad (k < \infty),
\]

\[
M^{(\infty)}(f) = \max |f - g|.
\]

Further a function \( f_0 \) in \( F \) will be called a best \( k \)-approximant to \( g \) if

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The existence and uniqueness of a best \( \infty \)-approximant was demonstrated in [5, Theorems 7 and 9].

**Theorem 1.** A best \( k \)-approximant always exists.

Let \( f_1, f_2, \ldots \) be a sequence of functions (in the \( n \)-parameter family \( F \)) such that \( \lim_{n \to \infty} M_n(f_j) = m \), where \( m = \text{g.l.b.} M_n(f) \). Let \( d = (b-a)/(n+1) \), and let \( x_j = a + jd \ (j = 1, 2, \ldots, n) \). On the interval \( x_j - d \leq x \leq x_j + d \), let the minimum of \( |f_i - g| \) occur at \( x_{ij} \) and set \( y_{ij} = f_i(x_{ij}) \). Since the sequence \( \{M_i(f_i)\} \ (i = 1, 2, \ldots) \) is bounded the set of values \( y_{ij} \) is bounded. Hence for each \( j \), the points \( (x_{ij}, y_{ij}) \) have a limit point \( (x_j, y_j) \). Hence there is a subsequence \( \{f_{i'}\} \) of the sequence \( \{f_i\} \) such that \( \lim_{i' \to \infty} (x_{ij}, y_{ij}) = (x_j, y_j) \ (j = 1, \ldots, n) \). Consequently, if \( f \) is the function in \( F \) through \( (x_1, y_1), \ldots, (x_n, y_n) \), \( \lim_{i' \to \infty} f_{i'} = f \) uniformly [5, Theorem 5, p. 460]; thus \( M_{i'}(f_{i'}) \to M_n(f) \), and so \( f \) is a best \( k \)-approximant.

**Theorem 2.** A best \( k \)-approximant is unique if \( F \) is linear and \( k > 1 \).

When \( k \) is an even integer, the proof of Pólya [3] for the special case of polynomials is also effective here.

In general we use the strict convexity of \( y = |x|^k \). If \( f_1 \neq f_2 \) and \( f = (f_1 + f_2)/2 \) then \( |f - g|^k \leq (|f_1 - g|^k + |f_2 - g|^k)/2 \), with inequality at some point; thus, since the functions are continuous \( M_n(f) < [M_n(f_1) + M_n(f_2)]/2 \). Hence the best approximant is unique. Were \( k = 1 \) all that we could say on the basis of this proof is that \( (f_1 - g)(f_2 - g) \geq 0 \) if \( f_1 \) and \( f_2 \) are best 1-approximants.

**Theorem 3.** If \( n = 1 \), a best \( k \)-approximant need not be unique for \( k \neq \infty \). Also it need not cross \( g \).

We give an example for \( k = 1 \). Let \( a = 0, b = 1 \). Let \( g(x) = 0 \). We designate by \( f_r \) that \( f \) in \( F \) for which \( f(0) = r \), i.e., by its \( y \)-intercept. Let \( f_r \) for \( r \leq -1 \) be the line segment of slope 1; for \( -1 \leq r \leq -1/2 \) the pair of line segments, one joining \((0, r)\) to \((s, s + r)\) having slope 1 and the other \((s, s + r)\) to \((1, 3 - 2s + r)\) having slope 3, where \( s = 3 + 2r - [3(r^2 + 3r + 2)]^{1/2} \); and for \(-1/2 \leq r \leq r, f_r = f_{-1} + r + 1/2 \). Then all \( f_r \) \(( -1 \leq r \leq -1/2 \) are best 1-approximants to \( g \) and \( f_{-1} \) does not cross \( g \). Only a slight change in the definition of \( s \) is needed to take care of other values of \( k \).

**Theorem 4.** Let \( f_k \) be a sequence of best \( k \)-approximants where \( k \to \infty \). Then \( \lim_{k \to \infty} f_k = f_\infty \).
The proof of Pólya [3] for the case of polynomials actually takes care of the general case after we have proved that there is a subsequence of the $f_n$ which approaches a limit $f^*$ uniformly. This can be done as in the proof of Theorem 1 after we have shown that $M^{(k)}(f_n)$ is bounded. But this fact was also proved by Pólya as follows:

$$M^{(k)}(f_n) \leq M^{(k)}(f_\infty) \leq [M^{(\infty)}(f_\infty)]^k(b - a) .$$

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