Let $X$ be a uniform space, let $T$ be a multiplicative topological group, and let $T$ act as a transformation group on $X$.

A subset $A$ of $T$ is said to be (left) syndetic provided that $T = AK$ for some compact subset $K$ of $T$. The transformation group $T$ is said to be almost periodic on $X$ provided that if $\alpha$ is an index of $X$, then there exists a syndetic subset $A$ of $T$ such that $xA \subseteq x\alpha$ for all $x \in X$. If $x \in X$, then the transformation group $T$ is said to be locally almost periodic at $x$ provided that if $U$ is a neighborhood of $x$, then there exists a neighborhood $V$ of $x$ and a syndetic subset $A$ of $T$ such that $VA \subseteq U$. The transformation group $T$ is said to be locally almost periodic on $X$ in case $T$ is locally almost periodic at every point of $X$.

If $x \in X$, then the transformation group $T$ is said to be locally weakly almost periodic at $x$ provided that if $U$ is a neighborhood of $x$, then there exists a neighborhood $V$ of $x$, a syndetic subset $A$ of $T$, and a compact subset $C$ of $T$ such that $y \in V$ implies the existence of a subset $B$ of $T$ for which $A \subseteq BC$ and $yB \subseteq U$. It is readily proved that if $x \in X$, then $T$ is locally weakly almost periodic at $x$ if and only if for each neighborhood $U$ of $x$ there exist a neighborhood $V$ of $x$ and a compact subset $K$ of $T$ such that $VT \subseteq UK$.

If $x \in X$, then the transformation group $T$ is said to be equicontinuous at $x$ provided that if $\alpha$ is an index of $X$, then there exists an index $\beta$ of $X$ such that $x\beta \subseteq xt\alpha$ for all $t \in T$. The transformation group $T$ is said to be equicontinuous on $X$ in case $T$ is equicontinuous at every point of $X$. The transformation group $T$ is said to be uniformly equicontinuous on $X$ provided that if $\alpha$ is an index of $X$, then there exists an index $\beta$ of $X$ such that $x\beta \subseteq xt\alpha$ for all $x \in X$ and all $t \in T$. It is readily verified that if $X$ is compact, then $T$ is uniformly equicontinuous if and only if $T$ is equicontinuous.

The transformation group $T$ is said to be distal on $X$ provided that if $x, y \in X$ with $x \neq y$, then there exists an index $\alpha$ of $X$ such that $(x, y) \notin x\alpha$ for all $t \in T$.

We also consider $T$ to be a transformation group acting on $X \times X$ in the following manner: if $x, y \in X$ and if $t \in T$, then $(x, y)t$ is defined to be $(xt, yt)$.

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As a general reference for the notions occurring in this paper, see [2].

**Lemma 1.** Let $X$ be compact and suppose there exists $x \in X$ such that $T$ is locally weakly almost periodic at $(x, x)$ but $T$ is not equicontinuous at $x$. Then $T$ is not distal on $X$.

**Proof.** Let $\mathcal{A}$ be the neighborhood filter of $x$. Since $T$ is not equicontinuous at $x$, there exists an open neighborhood $U$ of the diagonal of $X \times X$ such that $(N \times N)T \subseteq U$ for every $N \in \mathcal{A}$. Define $\mathcal{F} = \{(N \times N)T \cap U' : N \in \mathcal{A}\}$ where $U'$ is the complement of $U$ in $X \times X$. In order to show that $T$ is not distal on $X$ it is enough to show that $\mathcal{F} \neq \emptyset$. Since $\mathcal{F} = \{F : F \in \mathcal{F}\}$ is a closed filter-base on the compact set $U'$, it follows that $\bigcap_{\mathcal{F}} \neq \emptyset$. We complete the proof by showing that each member of $\mathcal{F}$ contains some member of $\mathcal{F}$ whence $\bigcap_{\mathcal{F}} \cap \mathcal{F}$ and thus $\bigcap_{\mathcal{F}} \neq \emptyset$. Let $N \in \mathcal{A}$. Choose a closed neighborhood $N_1$ of $x$ such that $N_1 \subseteq N$. Since $T$ is locally weakly almost periodic at $(x, x)$, there exist a neighborhood $M$ of $x$ and a compact subset $K$ of $T$ such that $(M \times M)T \subseteq (N_1 \times N_1)K$ whence $\operatorname{cls}(M \times M)T \subseteq (N_1 \times N_1)K \subseteq (N \times N)T \cap U'$. This shows that every member of $\mathcal{F}$ contains some number of $\mathcal{F}$. The proof is completed.

**Theorem 1.** Let $X$ be compact. Then the following statements are pairwise equivalent:

1. $T$ is almost periodic.
2. $T$ is locally almost periodic on $X$ and $T$ is distal on $X$.
3. $T$ is locally weakly almost periodic at every point of the diagonal of $X \times X$ and $T$ is distal on $X$.

**Proof.** It is known [2, 4.38] that if $X$ is compact, then $T$ is almost periodic if and only if $T$ is uniformly equicontinuous. It follows easily from this theorem that (1) implies (2). An independent proof that (1) implies (2) may also be given.

Suppose now that $T$ is locally almost periodic on $X$ and let $x \in X$. We show $T$ is locally weakly almost periodic at $(x, x)$. Let $U$ be a neighborhood of $x$. There exist a neighborhood $V$ of $x$ and a syndetic subset $A$ of $T$ such that $VA \subseteq U$. Let $K$ be a compact subset of $T$ for which $T = AK$. We conclude that $(V \times V)T \subseteq (U \times U)K$.

It is now clear that (2) implies (3). That (3) implies (1) is immediate from Lemma 1. The proof of the theorem is completed.

We say that $X$ is a *minimal orbit-closure under $T$* or simply that $X$ is *minimal under $T$* in case $xT = X$ for all $x \in X$. A discrete flow is a transformation group whose phase group $T$ is the additive group of
integers with the discrete topology. A discrete flow is completely determined by a homeomorphism of the phase space \( X \) onto itself.

None of the conditions in (2) or (3) of Theorem 1 is redundant. It is known [2, 12.63] that there exist compact metrizable zero-dimensional locally almost periodic minimal orbit-closures under discrete flows which are not almost periodic. A simple example of a ring of concentric circles rotating at different rates about their common center shows that distal alone does not imply almost periodicity.

Let \( x, y \in X \). The pair \((x, y)\) is said to be \textit{proximal under} \( T \) provided that if \( \alpha \) is an index of \( X \), then there exists \( t \in T \) such that \((xt, yt) \in \alpha\). Of course, there exists a pair of distinct points of \( X \) which is proximal under \( T \) if and only if \( T \) is not distal on \( X \). The pair \((x, y)\) is said to be \textit{syndetically proximal under} \( T \) provided that if \( \alpha \) is an index of \( X \), then there exists a syndetic subset \( A \) of \( T \) such that \((xa, ya) \in \alpha \) for all \( a \in A \).

**Lemma 2.** Let \( X \) be compact, let \( x, y \in X \), let \((x, y)\) be proximal under \( T \), let \( \alpha \) be an index of \( X \), and let \( K \) be a compact subset of \( T \). Then there exists \( t \in T \) such that \((xtk, ytk) \in \alpha \) for all \( k \in K \).

**Proof.** Since \( X \times K \) is compact, the phase projection \( \pi: X \times T \to X \) defined by \( \pi(x, t) = xt \) is uniformly continuous on \( X \times K \). Hence there exists an index \( \beta \) of \( X \) such that \((x_1, x_2) \in \beta \) implies \((x_1k, x_2k) \in \alpha \) for all \( k \in K \). Choose \( t \in T \) such that \((xt, yt) \in \beta \). The conclusion follows.

**Lemma 3.** Let \( X \) be compact, let \( x, y \in X \), and let \( T \) be locally almost periodic at \( x \). Then \((x, y)\) is syndetically proximal if and only if \((x, y)\) is proximal.

**Proof.** The necessity is obvious. We prove the sufficiency. Suppose \((x, y)\) is proximal. Let \( \alpha \) be an index of \( X \). Choose a neighborhood \( U \) of \( x \) such that \( U \times U \subseteq \alpha \). There exist a neighborhood \( V \) of \( x \) and a syndetic subset \( A \) of \( T \) such that \( VA \subseteq U \). Choose an index \( \beta \) of \( X \) such that \( xB \subseteq V \). There exists a syndetic subset \( B \) of \( T \) such that \( xB \subseteq \beta \). Let \( H \) be a compact subset of \( T \) such that \( T = BH \). By Lemma 2 there exists \( t_0 \in T \) such that \((xt_0h^{-1}, yt_0h^{-1}) \in \beta \) for all \( h \in H \). Now \( t_0 = b_0h_0 \) for some \( b_0 \in B \) and some \( h_0 \in H \). Since \( t_0h_0^{-1} = b_0 \), it follows that \((xb_0, yb_0) \in B \). Altogether we now have \( xb_0 \subseteq \beta \subseteq V \) and \( yb_0 \subseteq \beta \subseteq V \) whence \( xb_0, yb_0 \subseteq V \) and \((xb_0a, yb_0a) \subseteq U \times U \subseteq \alpha \) for all \( a \in A \). Thus \( t \in b_0A \) implies \((xt, yt) \in \alpha \). Since \( b_0A \) is a syndetic subset of \( T \), the proof is completed.

**Theorem 2.** Let \( X \) be compact and let \( T \) be locally almost periodic on \( X \). Then there exists a pair of distinct points of \( X \) which is syndetically proximal under \( T \) if and only if \( T \) is not equicontinuous on \( X \).
Proof. Use Theorem 1 and Lemma 3.

Let \( x, y \in X \). The pair \((x, y)\) is said to be \((\text{totally})\) asymptotic under \( T \) provided that if \( \alpha \) is an index of \( X \), then there exists a compact subset \( K \) of \( T \) such that \((x_t, y_t) \in \alpha\) for all \( t \in T - K \).

A study (see below) of the example in [1] will reveal that the phrase "syndetically proximal" in Theorem 2 cannot be replaced by "asymptotic," even though it is assumed in addition that \( X \) is a compact plane set, \( T \) is a discrete flow, \( X \) is minimal under \( T \), and \( T \) is regularly almost periodic at some points of \( X \).

We indicate briefly why this example cannot possess even a unilaterally asymptotic pair of distinct points. We adopt here the notation used in [1] and we assume familiarity with the paper. First of all, \( T \) is locally almost periodic since \( X \) is minimal under \( T \) and some points of \( X \) are regularly almost periodic. This is an application of a general theorem (cf. [2; 5.24]). Since \( f \) has equicontinuous powers, no two components of \( X \) are even proximal. Consequently, any asymptotic pair of points belonging to \( X \) would have to both lie in the same component. Now the nondegenerate components cannot have lengths which tend to zero under iteration of \( T \). This is shown as follows:

Let \( x \) be the point \((3^0, 3^0 + 3^1, 3^0 + 3^1 + 3^2, \ldots)\) of \( A \). The component \( V \) of \( X \) which lies over \( x \) is the longest and indeed has length 1. Let \( \alpha_0, \ldots, \alpha_n \) be an arbitrary finite sequence made up of 0, 1, 2. If the map \( f \) is applied \((\alpha_0 - 1)3^0 + (\alpha_1 - 1)3^1 + \cdots + (\alpha_n - 1)3^n \) times to \( x \), then the point \((\alpha_0 3^0, \alpha_0 3^0 + \alpha_1 3^1, \ldots)\) is obtained. Consequently the images of \( V \) constitute all nondegenerate components of \( X \) and are of length 1/2 infinitely often in both directions.

Bibliography


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