A hyperplane in a (real) topological linear space $E$ is a set of the form $f^{-1}c$, where $c$ is a real number and $f$ is a not-identically-zero continuous linear functional on $E$. The hyperplane determines two closed half-spaces—$f^{-1}[-\infty, c]$ and $f^{-1}[c, \infty]$], and two open half-spaces—$f^{-1}[-\infty, c[$ and $f^{-1}[c, \infty[$]. Subsets $A$ and $B$ of $E$ are said to be separated (strictly separated) by the hyperplane provided $A$ lies in one of the closed (open) half-spaces and $B$ in the other. An account of the principal known results on separation and strict separation of convex sets may be found in [1] and [3]. Our purpose here is to set forth some apparently new results on strict separation, including the following: If $A$ and $B$ are disjoint closed convex subsets of Euclidean $n$-space and neither contains a ray in its boundary, then $A$ and $B$ can be strictly separated by a hyperplane. A related problem was considered several years ago by the author [3, p. 459], but the present investigation was prompted by a question of Mr. Isaac Namioka.

The proof of our principal result (4) is based on several lemmas, of which the first is given below. A discussion of net-convergence (used below) is contained in [2]. The origin of the linear space under consideration will always be denoted by $\phi$.

(1) Suppose $C$ is a locally compact closed convex subset of a topological linear space, $\phi \in C$, $U$ is a neighborhood of $\phi$, and $S$ is a net in $C \setminus U$. Then there is a subnet $(x, \rightarrow)$ of $S$ and a corresponding net $(t, \rightarrow)$ of positive numbers such that $t_n$ converges to a number $t_0 < \infty$ and $t_n x_n$ converges to a point $y_0$ of $C \setminus \{\phi\}$.

PROOF. Since $C$ is closed and locally compact, $U$ contains a neighborhood $V$ of $\phi$ such that the intersection $B$ of $C$ with the boundary of $V$ is compact. For each member $k$ of the domain of $S$ we have $a_k S_k \subseteq B$ for some $a_k \in [0, 1]$. The desired conclusion then follows from the fact that $[0, 1]$ and $B$ are compact, and $\phi \not\in B$.

A subset of a linear space is linearly bounded provided its intersection with each line is bounded. For convex sets, this is equivalent to saying that the set contains no ray.

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735
(2) With $C$ as in (1), $C$ is compact if and only if $C$ is linearly bounded.

Proof. The "only if" part is trivial, and we wish to show that if $C$ is not compact, then $C$ contains a ray. Now if $C$ is not compact, there exist $S$ and $U$ as in (1) such that $S$ has no convergent subnet. And if $t_0 \neq 0$, $x_n$ converges to $t_0^{-1}y_0$, so it follows that $t_0 = 0$.

Let $y_n = t_n x_n$ for each $n$; then $[0, t_n^{-1}]y_n \subset C$, whence it follows that $[0, \infty \subset C$, completing the proof of (2).

(3) Suppose $A$ and $B$ are disjoint closed convex subsets of a topological linear space, and there exists a hyperplane $H$ separating $A$ from $B$ such that $A \cap H$ is nonempty and linearly bounded. Then if either $A$ or $B$ is locally compact, there is an open $V \ni \phi$ such that $A + V$ misses $B$.

Proof. Let the continuous linear functional $f$ and the real number $a$ be such that $H = f^{-1}[a, -\infty]$ and $A \subset f^{-1}[-\infty, a]$. Consider the case in which $A$ is locally compact. We wish first to show that the set $C = A \cap f^{-1}[a-1, a]$ is compact, and see by (1) that it suffices to show that $C$ is linearly bounded. Suppose $C$ contains a ray $T$ with initial point $x$, and let $y \in A \cap H$. From the fact that $A$ is closed and convex it follows readily that $A$ contains the ray $T' = y + (T-x)$. And since $T \subset f^{-1}[a-1, a]$, it is clear that $f$ is constant on $T$, whence $T' \subset A \cap H$, a contradiction showing that $C$ is compact. Now since $C$ is compact and disjoint from the closed set $S$, there exists open $U \ni \phi$ such that $C + U$ misses $B$. Let $V = U \cap f^{-1}[1/2, 1/2]$. Then $V$ is open, $\phi \in V$, $f^{-1}[-\infty, a-1] + V$ misses $B$, and $C + V$ misses $B$, so $V$ is the desired open set.

Consider now the case in which $B$ is locally compact. We may assume $\phi \in B$. Suppose the desired $V$ does not exist, and let $W$ be the set of all open $W \ni \phi$. Then there exists $U \in W$ such that for each $W \in W$ there are points $b \in B \setminus U$ and $a \in A$ with $b \in a + W$. Now let us apply (1), with the roles of $C$ and $S$ taken by $S$ and $(a, W \in W)$, respectively, to obtain nets $(x, >)$ and $(t, >)$ as described in (1), and let $(u, >)$ be the corresponding subnet of $(a, W \in W, \subset)$. If $t_0 > 0$, then $x_n$ converges to $t_0^{-1}y_0$, whence $u_n$ converges to $t_0^{-1}y_0$ and $A \cap B \neq \emptyset$, contradicting one of our hypotheses. Thus $t_0 = 0$, and as in (2), $[0, \infty \subset B$. Let $p \in A \cap H$ and for each $n$ let $z_n = t_n (u_n - p)$. Then $z_n$ converges to $y_0$ and always $p + [0, t_0^{-1}]z_n \subset A$, whence $p + [0, \infty \subset A$. Since the parallel rays $[0, \infty \ni y_0$ and $p + [0, \infty \ni y_0$ are separated by the hyperplane $H = f^{-1}a$, $f$ must be constant on each ray and it follows that $p + [0, \infty \ni y_0 \subset A \cap H$, a contradiction completing the proof of (3).

It is now easy to prove our principal result, which is
(4) For a locally compact closed convex subset $A$ of a locally convex topological linear space $E$, the following two assertions are equivalent: (i) Whenever $H$ is a hyperplane in $E$ which supports $A$, then $A \cap H$ is linearly bounded. (ii) Whenever $B$ is a closed convex subset of $E \setminus A$ which can be separated from $A$, then $B$ can be separated from $A$ by a hyperplane which misses $A$.

Proof. That (i) implies (ii) follows at once from (3), the local convexity of $E$, and the basic separation theorem for convex bodies. That (ii) implies (i) is obvious when $E$ is the plane, and general proof can be based on this case.

From (4) and the known special separation properties of finite-dimensional spaces, we have

(5) For a closed convex subset $A$ of $E^n$, the following two assertions are equivalent: (i) $A$ contains no ray in its boundary. (ii) Whenever $B$ is a closed convex subset of $E^n$ which misses $A$, then $B$ can be separated from $A$ by a hyperplane which misses $A$.

A partial analogue of (4) can be obtained by applying (3) for the case in which $B$ is locally compact. Other corollaries are obtained by observing that if $A$ and $B$ can be separated by two hyperplanes, one missing $A$ and the other missing $B$, then $A$ and $B$ can be strictly separated. (Together with (5), this yields the result stated in the introduction.) Another consequence of (5) is

(6) Suppose $A$ is a closed convex subset of $E^n$ and that $A$ contains no ray in its boundary. Then every linear manifold in $E^n$ which misses $A$ is contained in a hyperplane which misses $A$.

In conclusion, we remark that there are examples showing that the assumptions of local compactness cannot be avoided.

References


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