THE EXPANSION OF CERTAIN PRODUCTS

L. CARLITZ

1. Introduction. The familiar identities

\( \prod_{n=0}^{\infty} (1 + x^n) = \sum_{m=0}^{\infty} \frac{x^m}{m!} (m) \quad (|x| < 1), \)

\( \prod_{n=0}^{\infty} (1 - x^n)^{-1} = \sum_{m=0}^{\infty} \frac{1}{m!} (m) \quad (|x| < 1), \)

where

\( (x)_m = (1 - x)(1 - x^2) \cdots (1 - x^m), \quad (x_0) = 1, \)

suggest the problem of expanding the products

\( \prod_{m,n=0}^{\infty} (1 + x^m y^n), \quad \prod_{m,n=0}^{\infty} (1 - x^m y^n)^{-1} \quad (|x| < 1, |y| < 1). \)

Indeed Bellman \[1\] has recently proposed the expansion of the second of (1.4) as a research problem. We put

\( \prod_{m,n=0}^{\infty} (1 + x^m y^n), \quad \prod_{m,n=0}^{\infty} (1 - x^m y^n)^{-1} \quad (|x| < 1, |y| < 1). \)

In this note we derive some properties of \( G_m(x, y) \) and \( H_m(x, y) \) as well as some related functions.

2. To begin with we set up some recursion formulas satisfied by \( G_m \) and \( H_m \). Logarithmic differentiation of (1.5) yields

\[ \sum_{0}^{\infty} \frac{x^m y^n l}{1 + x^m y^n l} = \sum_{m=0}^{\infty} \frac{m l^m G_m(x, y)/(x)_m(y)_m}{\sum_{0}^{\infty} ml^m G_m/(x)_m(y)_m}. \]

Since the left member

\[ = \sum_{m,n=0}^{\infty} \sum_{r=1}^{\infty} (-1)^{r-1}(x^m y^n l)^r = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{l^r}{(1 - x^r)(1 - y^r)}, \]

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we get

\[ mG_m = \sum_{r=1}^{m} (-1)^{r-1} \frac{(x)_m(y)_m}{(x)_{m-r}(y)_{m-r}} G_{m-r} \frac{G_{m-r}}{(1 - x^r)(1 - y^r)}. \]

In exactly the same way logarithmic differentiation of (1.6) leads to

\[ mH_m = \sum_{r=1}^{m} \frac{(x)_m(y)_m}{(x)_{m-r}(y)_{m-r}} H_{m-r} \frac{H_{m-r}}{(1 - x^r)(1 - y^r)}. \]

It follows at once from (2.1) and (2.2) that \( G_m \) and \( H_m \) are polynomials in \( x \) and \( y \). We may easily compute the first few values using the recursion formulas:

\[
\begin{align*}
G_0 &= 1, \quad G_1 = 1, \quad G_2 = x + y, \\
G_3 &= xy + x^3 + x^2y + xy^2 + y^3 + x^2y^2, \\
H_0 &= 1, \quad H_1 = 1, \quad H_2 = 1 + xy, \\
H_3 &= 1 + xy + xy^2 + x^2y + x^2y^2 + x^3y^3.
\end{align*}
\]

Next if \( f(x) \) denotes the left member of (1.5), it is evident that

\[ f(x) = f(xt) \prod_{n=0}^{\infty} (1 + y^n t). \]

It follows from (2.3), (1.1), and (1.5) that

\[
\sum_{n=0}^{\infty} \frac{t^n G_m}{(x)_m(y)_m} = \sum_{n=0}^{\infty} \frac{x^n t^n G_m}{(x)_m(y)_m} \sum_{n=0}^{\infty} \frac{y^n (n-1)! t^n}{(y)_n}.
\]

Equating coefficients we get

\[
G_m = \sum_{r=0}^{m} x^{m-r} y^{r(r-1)/2} \frac{(x)_m}{(x)_{m-r}} \frac{(y)_m}{(y)_r(y)_{m-r}} G_{m-r},
\]

which yields

\[ G_m = \sum_{r=1}^{m} x^{m-r} y^{r(r-1)/2} \frac{(x)_{m-1}}{(x)_{m-r}} \left[ \begin{array}{c} m \\ r \end{array} \right] G_{m-r}, \]

where

\[ \left[ \begin{array}{c} m \\ r \end{array} \right] = \frac{(y)_m}{(y)_r(y)_{m-r}}. \]

The coefficients (2.5) occur in the identity

\[ \prod_{r=0}^{m-1} (1 + y^r t) = \sum_{r=0}^{m} y^{r(r-1)/2} \left[ \begin{array}{c} m \\ r \end{array} \right] t^r. \]
Interchanging $x$ and $y$ in (2.4) we have also

$$G_m = \sum_{r=1}^{m} x^{(r-1)/2} y^{m-r} \left[ \begin{array}{c} m \\ r \end{array} \right] \frac{(y)_{m-1}}{(y)_{m-r}} G_{m-r}.$$  

To get similar recursions for $H_m$, let $g(t)$ denote the left member of (1.6); then

$$g(t) = g(xt) \prod_{i=0}^{\infty} (1 - y^i t)^{-1}.$$  

Using (1.2) we get

$$H_m = \sum_{r=1}^{m} x^{m-r} \left[ \begin{array}{c} m \\ r \end{array} \right] \frac{(x)_{m-1}}{(x)_{m-r}} H_{m-r}$$

and on interchanging $x$ and $y$

$$H_m = \sum_{r=1}^{m} y^{m-r} \left[ \begin{array}{c} m \\ r \end{array} \right] \frac{(y)_{m-1}}{(y)_{m-r}} H_{m-r}.$$  

Since by (2.6),

$$\left[ \begin{array}{c} m \\ r \end{array} \right]$$

is a polynomial in $y$ with integral coefficients, it follows from (2.4) and (2.8) that $G_m$ and $H_m$ are polynomials in $x$ and $y$ with integral coefficients; this last fact is not evident from (2.1) and (2.2).

3. It is clear from (1.5) and (1.6) that

$$G_m(x, y) = G_m(y, x),$$

$$H_m(x, y) = H_m(y, x).$$

We now prove that

$$G_m(x, y) = (xy)^{(m-1)/2} G_m \left( \frac{1}{x}, \frac{1}{y} \right),$$

$$H_m(x, y) = (xy)^{(m-1)/2} H_m \left( \frac{1}{x}, \frac{1}{y} \right)$$

and

$$H_m(x, y) = y^{(m-1)/2} G_m \left( x, \frac{1}{y} \right).$$

We prove (3.2) by induction. In the right member of (2.1) replace
$x, y$ by $1/x, 1/y$; we get using the inductive hypothesis
\[
\sum_{r=1}^{m} (-1)^{r-1} \frac{(x)_m}{(x)_{m-r}} \frac{(y)_m}{(y)_{m-r}} \frac{G_{m-r}(x, y)}{(1 - x^r)(1 - y^r)} (xy)^{-k},
\]
where
\[
k = m(m + 1)/2 - (m - r)(m - r + 1)/2 - (m - r)(m - r - 1)/2 - r
\]
\[
= m(m + 1)/2 - (m - r) - r = m(m - 1)/2.
\]
This evidently completes the induction. In the same way we use (2.2) to prove (3.3). To prove (3.4) we replace $y$ by $1/y$ in the right member of (2.1). Then by the inductive hypothesis we get
\[
\sum_{r=1}^{m} \frac{(x)_m(y)_m}{(x)_{m-r}(y)_{m-r}} \frac{H_{m-r}(x, y)}{(1 - x^r)(1 - y^r)} y^{-m(m-1)/2} = m y^{-m(m-1)/2} H_m(x, y),
\]
by (2.2). This completes the proof of (3.4).

If we put
\[
\prod_{m, n=0}^{\infty} \frac{1 + x^m y^n t}{1 - x^m y^n t} = \sum_{m=0}^{\infty} K_m(x, y) t^m/(x)_m(y)_m,
\]
then by (1.5), (1.6) and (2.5)
\[
(3.6) \quad K_m = \sum_{r=0}^{m} \left[ \begin{array}{c} m \\ r \end{array} \right] \left[ \begin{array}{c} m \\ r \end{array} \right] G_r H_{m-r}.
\]
Thus $K_m$ is a polynomial in $x, y$ with integral coefficients. Moreover $K_m$ has the following symmetry properties
\[
(3.7) \quad K_m(x, y) = K_m(y, x),
\]
\[
(3.8) \quad K_m(x, y) = y^{m(m-1)/2} K_m\left(x, \frac{1}{y}\right);
\]
the latter is an immediate consequence of (3.4) and (3.6). Incidentally (3.5) implies the quadratic relation
\[
(3.9) \quad \sum_{r=0}^{m} (-1)^r \left[ \begin{array}{c} m \\ r \end{array} \right] \left[ \begin{array}{c} m \\ r \end{array} \right] K_r K_{m-r} = 0 \quad (m > 1).
\]

Logarithmic differentiation of the left member of (3.5) gives
\[
\sum_{m, n=0}^{\infty} \left( \frac{x^m y^n}{1 + x^m y^n t} + \frac{x^m y^n}{1 - x^m y^n t} \right) = 2 \sum_{m, n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(x^m y^n)^{2r+1} t^{2r}}{(1 - x^{2r+1})(1 - y^{2r+1})}.
\]
We accordingly get the recursion

\[ mK_m = 2 \sum_{0 \leq 2r < m} \frac{(x)_m(y)_m}{(x)_{m-2r-1}(y)_{m-2r-1}} \frac{K_{m-2r-1}}{(1 - x^{2r+1})(1 - y^{2r+1})}. \]

It evidently follows from (3.10) that for \( m \) even, \( K_m \) is divisible by \((1 + x^{m/2})(1 + y^{m/2})\); indeed if \( m = 2^k \), then \( K_m \) is divisible by

\[ \frac{1 - x^m}{1 - x^k} \frac{1 - y^m}{1 - y^k}. \]

By means of (3.10) we may compute the first few values of \( K_m \):

\[
\begin{align*}
K_0 &= 1, \quad K_1 = 2, \quad K_2 = 2(1 + x)(1 + y), \\
K_3 &= 2(1 + x)(1 + y)(1 + 2xy + x^2 + y^2 + x^2y^2), \\
K_4 &= (1 + x)(1 + x^2)(1 + y)(1 + y^2) \left\{ (1 - x)(1 - x^3)(1 - y)(1 - y^3) + 1 + 2xy + x^2 + y^2 + x^2y^2 \right\}.
\end{align*}
\]

Expanding \( K_4 \) we find that some of the coefficients are negative. Since the coefficients of

\[ \left[ \begin{array}{c} m \\ r \end{array} \right] 
\]

are non-negative it therefore follows from (3.6) that some of the coefficients of \( G_4 \) and \( H_4 \) are negative.

4. In (1.6) take \( y = 0 \). Comparison with (1.2) gives

\[ H_m(x, 0) = 1; \]

similarly from (1.1) we get

\[ G_m(x, 0) = x^{m(m-1)/2}. \]

In view of (3.2) and (3.3) it follows that \( G_m \) and \( H_m \) are of degree \( m(m-1)/2 \) in each of \( x \) and \( y \). As for \( K_m \), (3.6) implies

\[ K_m(x, 0) = \sum_{r=0}^{m} \left[ \begin{array}{c} m \\ r \end{array} \right] x^{m(m-1)/2} = \prod_{0}^{m-1} (1 + x^r) \]

by (2.6), and therefore \( K_m(x, y) \) is also of degree \( m(m-1)/2 \) in each of \( x \) and \( y \).

In the next place if we take \( y = 1 \) in (2.1) we get

\[ G_m(x, 1) = \frac{1 - x^m}{1 - x} G_{m-1}(x, 1), \]
which yields

\[(4.3)\hspace{1cm} G_m(x, 1) = \frac{(x)_m}{(1 - x)^m}.\]

If we put \(y = \zeta\), a primitive \(r\)th root of unity, where \(r | m\), (2.1) implies

\[mG_m(x, \zeta) = (-1)^{r-1}C_m \frac{(x)_m}{(1 - x^r)(x)_{m-r}} G_{m-r}(x, \zeta),\]

where

\[C_m = \frac{\binom{m}{r}}{(1 - y^r)(y)_{m-r}} = m.\]

It follows that

\[(4.4) G_m(x, \zeta) = (-1)^{m/r - m/r} \frac{(x)_m}{(1 - x^r)^{m/r}},\]

which reduces to (4.3) for \(r = 1\). The corresponding formula for \(H_m\) is

\[(4.5) H_m(x, \zeta) = \frac{(x)_m}{(1 - x^r)^{m/r}}.\]

For \(K_m\) it follows from (3.10) that

\[(4.6) K_m(x, \zeta) = 0 \quad (m \geq 2, r \text{ even}),\]

while

\[(4.7) K_m(x, \zeta) = 2^{m/r} \frac{(x)_m}{(1 - x^r)^{m/r}} \quad (r \text{ odd}).\]

We also note that when \(m\) is odd, (3.10) implies

\[K_m(x, -1) = \frac{1 - x^{m}}{1 - x} K_{m-1}(x, -1) = 0\]

by (4.6). Thus \(K_m(x, y)\) is divisible by \((1 + x)(1 + y)\) for all \(m \geq 2\).

The following consequence of (4.3) and (4.5) may be noted:

\[(4.8) G_m(1, 1) = m! = H_m(1, 1).\]

Since, as we have seen, \(G_m(x, y)\) is of degree \(m(m-1)/2\) in each of \(x\) and \(y\), the number of terms is \(O(m^4)\). Therefore it follows from (4.8) that at least one coefficient of \(G_m\) is unbounded; the same is also true of \(H_m\) and \(K_m\). Indeed by (4.7) we have

\[(4.9) K_m(1, 1) = 2^m \cdot m!.\]

If \(\zeta\) is a primitive \(r\)th root of unity and \(\eta\) a primitive \(s\)th root of unity, \(r | m, s | m\), then (4.4) implies the following curious property.
\[(4.10)\quad G_m(\eta, \zeta) = \begin{cases} 0 & (r \neq s), \\ \frac{1}{(-1)^{m-m/r} m^r/r(m/r)!} & (r = s). \end{cases} \]

Like results hold for \(H_m\) and \(K_m\).

5. If \(x = y\), (1.5) and (1.6) become

\[(5.1)\quad \prod_0^\infty \left(1 + x^m t \right)^{m+1} = \sum_0^\infty t^m G_m(x)/(x)^3_m, \]
\[(5.2)\quad \prod_0^\infty \left(1 - x^m t \right)^{-m-1} = \sum_0^\infty t^m H_m(x)/(x)^3_m, \]

where \(G_m(x) = G_m(x, x), H_m(x) = H_m(x, x)\). Thus (2.1) and (2.2) reduce to

\[(5.3)\quad m G_m(x) = \sum_{r=1}^m (-1)^{r-1} \left(\frac{(x)_m}{(1 - x^r)(x)^{m-r}}\right)^2 G_{m-r}(x), \]
\[(5.4)\quad m H_m(x) = \sum_{r=1}^m \left(\frac{(x)_m}{(1 - x^r)(x)^{m-r}}\right)^2 H_{m-r}(x), \]

and similarly for (2.4) and (2.8). Also (3.2) and (3.3) imply

\[(5.5)\quad G_m(x) = x^{m(m-1)} G_m \left(\frac{1}{x}\right), \quad H_m(x) = x^{m(m-1)} H_m \left(\frac{1}{x}\right). \]

Since \(H_m(0) = 1\), it follows that \(H_m(x)\) is of degree \(m(m-1)\). On the other hand using (2.4) we can show that \(G_m(x)\) is divisible by exactly \(x^{m-1}\) which implies that \(G_m(x)\) is of degree \((m-1)^2\) for \(m \geq 1\). We also note that (4.8) gives

\[(5.6)\quad G_m(1) = m! = H_m, \]

so that at least one coefficient of \(G_m(x)\) and of \(H_m(x)\) is unbounded.

In conclusion we remark that the left member of (5.2) may be compared with the generating function for unrestricted plane partitions [2; 3, p. 243]:

\[(5.7)\quad \prod_1^\infty \left(1 - x^m\right)^{-m} = \sum_0^\infty P(m) x^m. \]

References