THE EXPANSION OF CERTAIN PRODUCTS

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1. Introduction. The familiar identities

$$\prod_{n=0}^{\infty}(1+x^n t) = \sum_{m=0}^{\infty} x^{m(m-1)/2} t^m (x)_m \qquad (|x| < 1), \tag{1.1}$$

$$\prod_{n=0}^{\infty}(1-x^n t)^{-1} = \sum_{m=0}^{\infty} t^m/(x)_m \qquad (|x| < 1), \tag{1.2}$$

where

$$(x)_m = (1-x)(1-x^2)\cdots(1-x^m), \qquad (x_0) = 1, \tag{1.3}$$

suggest the problem of expanding the products

$$\prod_{m,n=0}^{\infty}(1+x^m y^n t), \qquad \prod_{m,n=0}^{\infty}(1-x^m y^n t)^{-1} \quad (|x| < 1, |y| < 1). \tag{1.4}$$

Indeed Bellman [1] has recently proposed the expansion of the second of (1.4) as a research problem. We put

$$\prod_{m,n=0}^{\infty}(1+x^m y^n t) = \sum_{m=0}^{\infty} t^m G_m(x, y)/(x)_m (y)_m, \tag{1.5}$$

$$\prod_{m,n=0}^{\infty}(1-x^m y^n t)^{-1} = \sum_{m=0}^{\infty} t^m H_m(x, y)/(x)_m (y)_m. \tag{1.6}$$

In this note we derive some properties of $G_m(x, y)$ and $H_m(x, y)$ as well as some related functions.

2. To begin with we set up some recursion formulas satisfied by G_m and H_m. Logarithmic differentiation of (1.5) yields

$$\sum_{0}^{\infty} \frac{x^m y^n t}{1+x^m y^n t} = \frac{\sum_{0}^{\infty} m t^m G_m/(x)_m (y)_m}{\sum_{0}^{\infty} t^m G_m/(x)_m (y)_m}.$$

Since the left member

$$= \sum_{m,n=0}^{\infty} \sum_{r=1}^{\infty} (-1)^{r-1}(x^m y^n t)^r = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{t^r}{(1-x^r)(1-y^r)},$$

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we get

$$(2.1)\qquad mG_m = \sum_{r=1}^{m} (-1)^{r-1} \frac{(x)_m(y)_m}{(x)_{m-r}(y)_{m-r}} \frac{G_{m-r}}{(1-x^r)(1-y^r)}.$$

In exactly the same way logarithmic differentiation of (1.6) leads to

$$(2.2)\qquad mH_m = \sum_{r=1}^{m} \frac{(x)_m(y)_m}{(x)_{m-r}(y)_{m-r}} \frac{H_{m-r}}{(1-x^r)(1-y^r)}.$$

It follows at once from (2.1) and (2.2) that G_m and H_m are polynomials in x and y. We may easily compute the first few values using the recursion formulas:

$$\begin{aligned} &G_0 = 1, \qquad G_1 = 1, \qquad G_2 = x + y, \\ &G_3 = xy + x^3 + x^2y + xy^2 + y^3 + x^2y^2, \\ &H_0 = 1, \qquad H_1 = 1, \qquad H_2 = 1 + xy, \\ &H_3 = 1 + xy + xy^2 + x^2y + x^2y^2 + x^3y^3. \end{aligned}$$

Next if $f(x)$ denotes the left member of (1.5), it is evident that

$$(2.3)\qquad f(x) = f(xt) \prod_{n=0}^{\infty} (1 + y^n t).$$

It follows from (2.3), (1.1), and (1.5) that

$$\sum_{0}^{\infty} \frac{t^m G^m}{(x)_m(y)_m} = \sum_{0}^{\infty} \frac{x^m t^m G_m}{(x)_m(y)_m} \sum_{0}^{\infty} \frac{y^{n(n-1)/2} t^n}{(y)_n}.$$

Equating coefficients we get

$$G_m = \sum_{r=0}^{m} x^{m-r} y^{r(r-1)/2} \frac{(x)_m}{(x)_{m-r}} \frac{(y)_m}{(y)_r(y)_{m-r}} G_{m-r},$$

which yields

$$(2.4)\qquad G_m = \sum_{r=1}^{m} x^{m-r} y^{r(r-1)/2} \frac{(x)_{m-1}}{(x)_{m-r}} \begin{bmatrix} m \\ r \end{bmatrix}_y G_{m-r},$$

where

$$(2.5)\qquad \begin{bmatrix} m \\ r \end{bmatrix}_y = \frac{(y)_m}{(y)_r(y)_{m-r}}.$$

The coefficients (2.5) occur in the identity

$$(2.6)\qquad \prod_{r=0}^{m-1} (1 + y^r t) = \sum_{r=0}^{m} y^{r(r-1)/2} \begin{bmatrix} m \\ r \end{bmatrix}_y t^r.$$

Interchanging x and y in (2.4) we have also

$$G_m = \sum_{r=1}^{m} x^{r(r-1)/2} y^{m-r} \begin{bmatrix} m \\ r \end{bmatrix}_x \frac{(y)_{m-1}}{(y)_{m-r}} G_{m-r}. \tag{2.7}$$

To get similar recursions for H_m, let $g(t)$ denote the left member of (1.6); then

$$g(t) = g(xt) \prod_0^\infty (1 - y^n t)^{-1}.$$

Using (1.2) we get

$$H_m = \sum_{r=1}^{m} x^{m-r} \frac{(x)_{m-1}}{(x)_{m-r}} \begin{bmatrix} m \\ r \end{bmatrix}_y H_{m-r} \tag{2.8}$$

and on interchanging x and y

$$H_m = \sum_{r=1}^{m} y^{m-r} \begin{bmatrix} m \\ r \end{bmatrix}_y \frac{(y)_{m-1}}{(y)_{m-1}} H_{m-r}. \tag{2.9}$$

Since by (2.6),

$$\begin{bmatrix} m \\ r \end{bmatrix}_y$$

is a polynomial in y with integral coefficients, it follows from (2.4) and (2.8) that G_m and H_m are polynomials in x and y with integral coefficients; this last fact is not evident from (2.1) and (2.2).

3. It is clear from (1.5) and (1.6) that

$$\begin{aligned} G_m(x, y) &= G_m(y, x), \\ H_m(x, y) &= H_m(y, x). \end{aligned} \tag{3.1}$$

We now prove that

$$G_m(x, y) = (xy)^{m(m-1)/2} G_m\left(\frac{1}{x}, \frac{1}{y}\right), \tag{3.2}$$

$$H_m(x, y) = (xy)^{m(m-1)/2} H_m\left(\frac{1}{x}, \frac{1}{y}\right) \tag{3.3}$$

and

$$H_m(x, y) = y^{m(m-1)/2} G_m\left(x, \frac{1}{y}\right). \tag{3.4}$$

We prove (3.2) by induction. In the right member of (2.1) replace

x, y by $1/x$, $1/y$; we get using the inductive hypothesis

$$\sum_{r=1}^{m}(-1)^{r-1}\frac{(x)_m}{(x)_{m-r}}\frac{(y)_m}{(y)_{m-r}}\frac{G_{m-r}(x,y)}{(1-x^r)(1-y^r)}(xy)^{-k},$$

where

$$\begin{aligned}k &= m(m+1)/2-(m-r)(m-r+1)/2-(m-r)(m-r-1)/2-r\\ &= m(m+1)/2-(m-r)-r = m(m-1)/2.\end{aligned}$$

This evidently completes the induction. In the same way we use (2.2) to prove (3.3). To prove (3.4) we replace y by $1/y$ in the right member of (2.1). Then by the inductive hypothesis we get

$$\sum_{r=1}^{m}\frac{(x)_m(y)_m}{(x)_{m-r}(y)_{m-r}}\frac{H_{m-r}(x,y)}{(1-x^r)(1-y^r)}y^{-m(m-1)/2} = my^{-m(m-1)/2}H_m(x,y),$$

by (2.2). This completes the proof of (3.4).

If we put

$$\prod_{m,n=0}^{\infty}\frac{1+x^my^nt}{1-x^my^nt} = \sum_{m=0}^{\infty}K_m(x,y)t^m/(x)_m(y)_m, \tag{3.5}$$

then by (1.5), (1.6) and (2.5)

$$K_m = \sum_{r=0}^{m}\begin{bmatrix}m\\r\end{bmatrix}_x\begin{bmatrix}m\\r\end{bmatrix}_y G_rH_{m-r}. \tag{3.6}$$

Thus K_m is a polynomial in x, y with integral coefficients. Moreover K_m has the following symmetry properties

$$K_m(x,y) = K_m(y,x), \tag{3.7}$$

$$K_m(x,y) = y^{m(m-1)/2}K_m\left(x,\frac{1}{y}\right); \tag{3.8}$$

the latter is an immediate consequence of (3.4) and (3.6). Incidentally (3.5) implies the quadratic relation

$$\sum_{r=0}^{m}(-1)^r\begin{bmatrix}m\\r\end{bmatrix}_x\begin{bmatrix}m\\r\end{bmatrix}_y K_rK_{m-r} = 0 \qquad (m>1). \tag{3.9}$$

Logarithmic differentiation of the left member of (3.5) gives

$$\begin{aligned}\sum_{m,n=0}^{\infty}\left(\frac{x^my^n}{1+x^my^nt}+\frac{x^my^n}{1-x^my^nt}\right) &= 2\sum_{m,n=0}^{\infty}\sum_{r=0}^{\infty}(x^my^n)^{2r+1}t^{2r}\\ &= 2\sum_{r=0}^{\infty}\frac{t^{2r}}{(1-x^{2r+1})(1-y^{2r+1})}.\end{aligned}$$

We accordingly get the recursion

$$(3.10)\quad mK_m = 2\sum_{0\leq 2r<m} \frac{(x)_m(y)_m}{(x)_{m-2r-1}(y)_{m-2r-1}} \frac{K_{m-2r-1}}{(1-x^{2r+1})(1-y^{2r+1})}.$$

It evidently follows from (3.10) that for m even, K_m is divisible by $(1+x^{m/2})(1+y^{m/2})$; indeed if $m=2^e k$, then K_m is divisible by

$$\frac{1-x^m}{1-x^k}\,\frac{1-y^m}{1-y^k}.$$

By means of (3.10) we may compute the first few values of K_m:

$$K_0 = 1, \qquad K_1 = 2, \qquad K_2 = 2(1+x)(1+y),$$
$$K_3 = 2(1+x)(1+y)(1+2xy+x^2+y^2+x^2y^2),$$
$$K_4 = (1+x)(1+x^2)(1+y)(1+y^2)\{(1-x)(1-x^2)(1-y)(1-y^2) + 1 + 2xy + x^2 + y^2 + x^2y^2\}.$$

Expanding K_4 we find that some of the coefficients are negative. Since the coefficients of

$$\begin{bmatrix} m \\ r \end{bmatrix}_x$$

are non-negative it therefore follows from (3.6) that some of the coefficients of G_4 and H_4 are negative.

4. In (1.6) take $y=0$. Comparison with (1.2) gives

$$(4.1)\qquad H_m(x, 0) = 1;$$

similarly from (1.1) we get

$$(4.2)\qquad G_m(x, 0) = x^{m(m-1)/2}.$$

In view of (3.2) and (3.3) it follows that G_m and H_m are of degree $m(m-1)/2$ in each of x and y. As for K_m, (3.6) implies

$$K_m(x, 0) = \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix}_x x^{m(m-1)/2} = \prod_0^{m-1}(1+x^r)$$

by (2.6), and therefore $K_m(x, y)$ is also of degree $m(m-1)/2$ in each of x and y.

In the next place if we take $y=1$ in (2.1) we get

$$G_m(x, 1) = \frac{1-x^m}{1-x} G_{m-1}(x, 1),$$

which yields

$$G_m(x, 1) = (x)_m/(1 - x)^m \cdot \tag{4.3}$$

If we put $y=\zeta$, a primitive rth root of unity, where $r|m$, (2.1) implies

$$mG_m(x, \zeta) = (-1)^{r-1}C_m \frac{(x)_m}{(1 - x^r)(x)_{m-r}} G_{m-r}(x, \zeta),$$

where

$$C_m = \left\{\frac{(y)_m}{(1 - y^r)(y)_{m-r}}\right\}_{y=\zeta} = m.$$

It follows that

$$G_m(x, \zeta) = (-1)^{m-m/r} \frac{(x)_m}{(1 - x^r)^{m/r}}, \tag{4.4}$$

which reduces to (4.3) for $r=1$. The corresponding formula for H_m is

$$H_m(x, \zeta) = (x)_m/(1 - x^r)^{m/r}. \tag{4.5}$$

For K_m it follows from (3.10) that

$$K_m(x, \zeta) = 0 \qquad (m \geqq 2,\ r \text{ even}), \tag{4.6}$$

while

$$K_m(x, \zeta) = 2^{m/r} \frac{(x)_m}{(1 - x^r)^{m/r}} \qquad (r \text{ odd}). \tag{4.7}$$

We also note that when m is odd, (3.10) implies

$$K_m(x, -1) = \frac{1 - x^m}{1 - x} K_{m-1}(x, -1) = 0$$

by (4.6). Thus $K_m(x, y)$ is divisible by $(1+x)(1+y)$ for all $m \geqq 2$.

The following consequence of (4.3) and (4.5) may be noted:

$$G_m(1, 1) = m! = H_m(1, 1). \tag{4.8}$$

Since, as we have seen, $G_m(x, y)$ is of degree $m(m-1)/2$ in each of x and y, the number of terms is $O(m^4)$. Therefore it follows from (4.8) that at least one coefficient of G_m is unbounded; the same is also true of H_m and K_m. Indeed by (4.7) we have

$$K_m(1, 1) = 2^m \cdot m!. \tag{4.9}$$

If ζ is a primitive rth root of unity and η a primitive sth root of unity, $r|m$, $s|m$, then (4.4) implies the following curious property.

$$G_m(\eta, \zeta) = \begin{cases} 0 & (r \neq s), \\ (-1)^{m-m/r} r^{m/r} (m/r)! & (r = s). \end{cases} \tag{4.10}$$

Like results hold for H_m and K_m.

5. If $x=y$, (1.5) and (1.6) become

$$\prod_0^\infty (1 + x^m t)^{m+1} = \sum_0^\infty t^m G_m(x)/(x)_m^2, \tag{5.1}$$

$$\prod_0^\infty (1 - x^m t)^{-m-1} = \sum_0^\infty t^m H_m(x)/(x)_m^2, \tag{5.2}$$

where $G_m(x) = G_m(x, x)$, $H_m(x) = H_m(x, x)$. Thus (2.1) and (2.2) reduce to

$$mG_m(x) = \sum_{r=1}^m (-1)^{r-1} \left\{ \frac{(x)_m}{(1 - x^r)(x)_{m-r}} \right\}^2 G_{m-r}(x), \tag{5.3}$$

$$mH_m(x) = \sum_{r=1}^m \left\{ \frac{(x)_m}{(1 - x^r)(x)_{m-r}} \right\}^2 H_{m-r}(x), \tag{5.4}$$

and similarly for (2.4) and (2.8). Also (3.2) and (3.3) imply

$$G_m(x) = x^{m(m-1)} G_m\left(\frac{1}{x}\right), \qquad H_m(x) = x^{m(m-1)} H_m\left(\frac{1}{x}\right). \tag{5.5}$$

Since $H_m(0) = 1$, it follows that $H_m(x)$ is of degree $m(m-1)$. On the other hand using (2.4) we can show that $G_m(x)$ is divisible by exactly x^{m-1} which implies that $G_m(x)$ is of degree $(m-1)^2$ for $m \geqq 1$. We also note that (4.8) gives

$$G_m(1) = m! = H_m, \tag{5.6}$$

so that at least one coefficient of $G_m(x)$ and of $H_m(x)$ is unbounded.

In conclusion we remark that the left member of (5.2) may be compared with the generating function for unrestricted plane partitions [**2**; **3**, p. 243]:

$$\prod_1^\infty (1 - x^m)^{-m} = \sum_0^\infty P(m) x^m. \tag{5.7}$$

References

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