ON AN INTEGRO-DIFFERENTIAL OPERATOR OF THE CAUCHY TYPE[1]

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1. **Introduction.** The theory of the Cauchy process leads to the following integro-differential equation:

$$(1.1) \qquad u_t(x, t) = \pi^{-1}P\int_{-a}^{+a} \frac{u_\xi(\xi, t)}{\xi - x}\, d\xi, \qquad -\infty < -a < x < a < \infty,$$

the integral being taken in the sense of a Cauchy principal value. It is shown in [3] that (1.1) with appropriate boundary conditions defines stochastic processes which bear much the same relation to the Cauchy process as the diffusion processes associated with

$$(1.2) \qquad u_t = u_{xx}, \qquad -\infty < -a \leqq x \leqq a < \infty,$$

bear to the Wiener process.

If initial values $u(x, 0) = f(x)$ and boundary conditions are prescribed for (1.1) (see [2] for the types of boundary conditions which may be imposed), then the familiar method of Laplace transforms commonly employed for solving partial differential equations leads formally to the so-called resolvent equation

$$(1.3) \qquad \lambda F(x) - \pi^{-1}P\int_{-a}^{+a} \frac{F'(\xi)}{\xi - x}\, d\xi = f(x), \qquad \lambda > 0,$$

where

$$(1.4) \qquad F(x) = \int_0^\infty e^{-\lambda t} u(x, t)\, dt;$$

the boundary conditions for (1.1) go over into boundary conditions for (1.4). An equation essentially equivalent to (1.3) was derived by Kac in [4] and treated by Kac and Pollard in [5]. Boundary conditions and construction of solutions to (1.1) were considered in [2]. Equation (1.3) with boundary conditions $u(-a, t) = u(a, t) = 0$ is well-known in airfoil theory as a special case of the Prandtl equation, cf. [6].

For $\lambda > 0$, the resolvent equation (1.3) can be solved by reducing it to an integral equation of the Fredholm type, cf. [2] and [6],

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whose solution is given by an integral of the form

$$F(x) = \int_{-a}^{+a} \Gamma_a(x, y; \lambda) f(y) dy. \tag{1.5}$$

It has been shown in [2] that Γ_a is non-negative, symmetric in x and y, and that (1.5) defines a bounded linear transformation from $C[-a, +a]$ to itself. Furthermore, if $f \in C[-a, +a]$, then F is a solution of (1.3) which satisfies

$$\lim_{x \to \pm a} F(x) = 0. \tag{1.6}$$

In this note we shall be interested in (1.5) when f is not necessarily continuous on $[-a, +a]$, but is of the form $\phi(x)(a^2-x^2)^{-1}$ where $\phi \in C[-a, +a]$. We shall denote this class of functions by U. The results which we obtain here have many applications in the theory of the Cauchy and allied stochastic processes, and are employed in this connection in [3].

In §3, we show that if $f \in U$, then F as defined by (1.5) is an element of $C[-a, +a]$. We also show that the solutions of the homogeneous equation corresponding to (1.3), i.e. (1.3) with $f \equiv 0$, can be expressed in the form (1.5) with $f \in U$.

It might be supposed that F still satisfies (1.3) when f is in U instead of being restricted to C. In §4, we show that this is not true in general and we derive the functional equation satisfied by F in the more general case. This new functional equation is used in [3] to derive the forward and backward equations for the stochastic processes discussed there, and also to justify probabilistically the interpretation of (1.1) as the backward equation corresponding to the Cauchy process in a finite interval.

2. **Preliminaries.** In this section we collect a few known results to which we refer frequently in the sequel.

DEFINITION 2.1. *If F is absolutely continuous on $[-a, +a]$, $[F'(x)]^2(a^2-x^2)^{1/2} \in [-a, +a]$ and*[2]

$$\pi^{-1} P \int_{-a}^{+a} \frac{F'(t)}{t-x} dt = h(x) \tag{2.1}$$

almost everywhere on $[-a, +a]$ with $h \in C[-a, +a]$, then we define

[2] The condition $[F'(x)]^2(a^2-x^2) \in L(-a, a)$ should have been included in [2, Definition 1.2]. Whether this condition is necessary for the reduction of (2.3) to (2.7) is an open question. If (2.4) with $g(x) \equiv 0$ has no solution $G \in L(-a, a)$, then the absolute continuity of F on $(-a, a)$ suffices for the equivalence of (2.3) and (2.7).

$$\Omega_a F = h \tag{2.2}$$

as a transformation from $C[-a, +a]$ *to itself.*

We now make our statements about (1.3) more precise. Given $f \in C[-a, +a]$, consider the equation

$$\lambda F - \Omega_a F = f. \tag{2.3}$$

The main result which enables us to reduce (2.3) to an integral equation is the following inversion formula due to Söhngen (see [2] and [6] for reference).

THEOREM 2.1. *If* $g^2(x)(a^2-x^2)^{1/2} \in L[-a, +a]$, *then the most general solution* G *of*

$$\pi^{-1} P \int_{-a}^{+a} \frac{G(u)}{u-x}\, du = g(x) \qquad a.e. \tag{2.4}$$

for $x \in [-a, +a]$, *such that* $G^2(u)(a^2-u^2)^{1/2} \in L[-a, +a]$, *is given by*

$$G(u) = (a^2 - u^2)^{-1/2} \left[A - \pi^{-1} P \int_{-a}^{+a} \frac{g(x)(a^2 - x^2)^{1/2}}{x - u}\, dx \right] \tag{2.5}$$

almost everywhere in $[-a, +a]$, *where*

$$A = \pi^{-1} \int_{-a}^{+a} G(u) du. \tag{2.6}$$

Applying Theorem 2.1 to (2.3) with $f \in C[-a, +a]$, we obtain an equivalent integral equation, cf. [2] and [6],

$$F(x) + \lambda \int_{-a}^{+a} K_a(x, y) F(y) dy = \int_{-a}^{+a} K_a(x, y) f(y) dy + A \arcsin x + B \tag{2.7}$$

where

$$K_a(x, y) = (2\pi)^{-1} \log \left[\frac{a^2 - xy + \{(a^2 - x^2)(a^2 - y^2)\}^{1/2}}{a^2 - xy - \{(a^2 - x^2)(a^2 - y^2)\}^{1/2}} \right]. \tag{2.8}$$

If (2.3) is subject to the boundary condition $F(-a) = F(a) = 0$, then $A = B = 0$ in (2.7). To find F under these boundary conditions, we make use of the resolvent kernel $\Gamma_a(x, y; \lambda)$ corresponding to $K_a(x, y)$, which satisfies

$$\Gamma_a(x, y; \lambda) + \lambda \int_{-a}^{+a} K_a(x, u) \Gamma_a(u, y; \lambda) du = K_a(x, y). \tag{2.9}$$

The solution F is then given by (1.5). This is, in fact, the only solution in $C[-a, +a]$ of (2.3) under the boundary condition (1.6). It was shown in [2, Theorem 2.1], that for each $\lambda>0$ the equation $\lambda F-\Omega_a F=0$ possesses two independent solutions in $C[-a, +a]$, given by

$$\xi_a(x) = 1 - \lambda \int_{-a}^{+a} \Gamma_a(x, y; \lambda)dy \tag{2.10}$$

and

$$\eta_a(x) = \pi^{-1} \arcsin x/a - \lambda\pi^{-1} \int_{-a}^{+a} \Gamma_a(x, y; \lambda) \arcsin y/a dy. \tag{2.11}$$

All other solutions of the homogeneous equation are linear combinations of these two. Since $\lim_{x\to\pm a} \xi_a(x)=1$ and $\lim_{x\to\pm a} \eta_a(x)=\pm 1/2$, it is clear that the boundary condition (1.6) cannot be satisfied by a solution of the homogeneous equation. We also proved in [2] that

$$0 \leqq \xi_a(x) \leqq 1, \tag{2.12}$$

which will prove useful in §3 where we shall derive slightly simpler formulas for ξ_a and η_a.

We conclude this section by stating two results for use in later sections.

THEOREM 2.2. *For each $p>1$, the transformation $G\to g$ defined by* (2.4) *is a bounded linear transformation from the space $L_p(-a, +a)$ to itself. It is also a bounded linear transformation from the space of functions G such that $G^2(x)(a^2-x^2)^{1/2}\in L[-a, +a]$ to itself.*

PROOF. The proof of the first statement may be found in [7, Chap. V, Theorem 101].

There are a number of ways of proving the second statement. We sketch one possible proof. We may rewrite $G(u)$ as

$$\begin{aligned} 2aG(u) = G(u)(a^2 - u^2)^{1/2}\{(a - u)^{1/2}(a + u)^{-1/2} \\ + (a - u)^{-1/2}(a + u)^{1/2}\} = H(u)\{w(u) + [w(u)]^{-1}\}. \end{aligned} \tag{2.13}$$

Now $H^2(u)w(u)$ and $H^2(u)[w(u)]^{-1}$ are both in $L[-a, +a]$. The result then follows easily from [1, Lemma 2].

LEMMA 2.1. *For each integer $n>0$*

$$\pi^{-1}P \int_{-a}^{+a} \frac{t^n}{(t - x)(a^2 - t^2)^{1/2}} dt = \sum_{k=1}^{n} \alpha_k x^{n-k} \tag{2.14}$$

where

$$\alpha_n = 0, \qquad n \text{ even},$$

$$\alpha_n = a^{n-1} \frac{1\cdot 3\cdot 5 \cdots (n-2)}{2\cdot 4\cdot 6 \cdots (n-1)}, \qquad n \text{ odd}, n>1 \tag{2.15}$$

$$\alpha_1 = 1.$$

PROOF. This is a well-known formula, frequently used in airfoil theory. A proof may be found in [2], in the proof of Lemma 1.1.

3. **Solutions to the homogeneous equation and functions of class** U. In this section we shall derive formulas for ξ_a and η_a (defined in (2.10) and (2.11)) in terms of functions of class U which we may define as follows:

DEFINITION 3.1. *By U we denote the Banach space of functions of the form*

$$f(x) = \phi(x)\cdot(a^2 - x^2)^{-1} \tag{3.1}$$

with $\phi \in C[-a, +a]$ and

$$\|f\|_U = \|\phi\|_C. \tag{3.2}$$

These formulas are of interest in themselves, cf. [3], and at the same time are useful for proving further properties of the transformation (1.5) when f is of the form (3.1).

THEOREM 3.1. *Defining ξ_a and η_a as in* (2.10) *and* (2.11), *we have*

$$\xi_a(x) = 2a\pi^{-1}\int_{-a}^{+a} \frac{\Gamma_a(x, y; \lambda)}{a^2 - y^2}\, dy, \tag{3.3}$$

$$\eta_a(x) = \pi^{-1}\int_{-a}^{+a} \frac{y\Gamma_a(x, y; \lambda)}{a^2 - y^2}\, dy. \tag{3.4}$$

Before proving this theorem, we require a lemma:

LEMMA 3.1. *If*

$$\Phi(x) = \int_{-a}^{+a} \frac{K_a(x, y)\phi(y)}{a^2 - y^2}\, dy \tag{3.5}$$

with $\phi \in C[-a, +a]$, then Φ is absolutely continuous on any interval $[x_1, x_2] \subset (-a, +a)$ and Φ' is given by

$$\Phi'(x) = [\pi(a^2 - x^2)^{1/2}]^{-1}\cdot P\int_{-a}^{+a} \frac{\phi(y)(a^2 - y^2)^{-1/2}}{y - x}\, dy \tag{3.6}$$

a.e. on $[-a, +a]$.

PROOF. Denote the function defined by the right-hand side of (3.6) by $G(x)$. From Theorem 2.2 it follows that G is integrable over any subinterval $[x_1, x_2] \subset (-a, +a)$. It is sufficient to prove then that for $x \in (-a, +a)$

$$\Phi(x) - \Phi(x_1) = \int_{x_1}^{x} G(u)du. \tag{3.7}$$

This can be proved by the method used for a similar theorem in [**2**, Lemma 3.2]. Since the details are almost identical, we omit the proof here.

We now go on to

PROOF OF THEOREM 3.1.

(a) *Proof of* (3.1). Because of relation (2.9) and the symmetry of Γ_a and K_a, all we need show is that

$$A(x) = 2a\pi^{-1} \int_{-a}^{+a} \frac{K_a(x, y)}{a^2 - y^2} dy \equiv 1, \tag{3.8}$$

for $x \in (-a, +a)$. From Lemma 3.1, (2.14), and (2.15) we see that $A(x) \equiv C$ for $x \in (-a, +a)$ where C is a constant. Now we make use of the formula

$$\int_{-a}^{+a} K_a(x, y)dy = (a^2 - x^2)^{1/2}, \tag{3.9}$$

which follows from [**2**, Lemma 5.1], to conclude that

$$\int_{-a}^{+a} A(x)dx = 2aC = 2a \quad \text{and} \quad \text{hence} \quad C = 1.$$

(b) *Proof of* (3.2). Again by (2.9), it is sufficient to prove that:

$$B(x) = \pi^{-1} \int_{-a}^{+a} \frac{yK_a(x, y)}{a^2 - y^2} dy = \pi^{-1} \arcsin x/a. \tag{3.10}$$

As in (a), the result follows from Lemma 3.1, (2.14), and (2.15).

The function Φ defined by (3.5) is continuous at each point of $(-a, +a)$, as we have shown in Lemma 3.1. The next theorem shows that we can define $\Phi(x)$ at $x = \pm a$, so that it is continuous on the closed interval $[-a, +a]$, and gives $\Phi(\pm a)$ in terms of $\phi(\pm a)$.

THEOREM 3.2. *If* $\Phi(x)$ *is defined as in Lemma* 3.1, *then*

$$\lim_{x \to \pm a} \Phi(x) = \pi(2a)^{-1}\phi(\pm a). \tag{3.11}$$

PROOF. We shall carry through the proof for $x \to a$. By (3.8), we have

$$(3.12)\quad 2a\Phi(x) - \pi\phi(a) = 2a \int_{-a}^{+a} K_a(x, y) \left[\frac{\phi(y) - \phi(a)}{a^2 - y^2} \right] dy.$$

The integral on the right can be split into two parts $I_{+1} + I_{-1}$ where

$$(3.13)\qquad I_{\pm 1} = \int_{-a}^{+a} K_a(x, y) \left[\frac{\phi(y) - \phi(a)}{a \pm y} \right] dy.$$

Now it follows from (3.8) and (3.10) that

$$(3.14)\qquad | I_{+1}(x) | \leqq 2\|\phi\|_C \{\pi/2 - \arcsin x/a\}.$$

Hence $I_{+1}(x) \to 0$ as $x \to a$.

Next, given $\epsilon > 0$, we can choose $\delta > 0$ so that $|\phi(y) - \phi(a)| < \epsilon\pi^{-1}$ when $0 < a - y < \delta$. The integral I_{-1} can be split into the two parts

$$(3.15)\qquad I_{-1} = \int_{-a}^{a-\delta} + \int_{a-\delta}^{a} = J_1 + J_2.$$

Using (3.9), we have

$$(3.16)\qquad | J_1(x) | \leqq 2\|\phi\|_C \delta^{-1}(a^2 - x^2)^{1/2}$$

and therefore $J_1(x) \to 0$ as $x \to a$. Finally, by (3.8) and (3.10)

$$(3.17)\qquad | J_2(x) | \leqq \epsilon\pi^{-1} \int_{-a}^{+a} \frac{K_a(x, y)}{a - y} dy \leqq \epsilon.$$

Results similar to those obtained in Lemma 3.1 and Theorem 3.2 can be proved with K_a replaced by Γ_a, namely

COROLLARY 3.1. *If $F(x)$ is defined by*

$$(3.18)\qquad F(x) = \int_{-a}^{+a} \frac{\Gamma_a(x, y; \lambda)}{a^2 - y^2} \phi(y) dy$$

with $\phi \in C[-a, +a]$, then F is absolutely continuous on any interval $[x_1, x_2] \subset (-a, +a)$. Furthermore

$$(3.19)\qquad \lim_{x \to \pm a} F(x) = \pi(2a)^{-1}\phi(\pm a).$$

PROOF. From (2.9), using the fact that Γ_a and K_a are symmetric in x and y, we have

$$(3.20)\qquad F(x) + \lambda \int_{-a}^{+a} \Gamma_a(x, y; \lambda)\Phi(y) dy = \Phi(x),$$

where Φ is defined as in Lemma 3.1. Now $\Phi \in C[-a, +a]$ and by [2, Theorems 2.1, 4.1 and 4.2] it follows that the integral in (3.20) defines an absolutely continuous function on $[-a, +a]$. Our result follows from Lemma 3.1, Theorem 3.2 and the fact that

$$\lim_{x \to \pm a} \int_{-a}^{+a} \Gamma_a(x, y; \lambda)\Phi(y)dy = 0. \tag{3.21}$$

We make the final remark that it follows easily from the results of this section and (2.12) that the linear transformations defined by

$$T_1 f = \int_{-a}^{+a} K_a(x, y)f(y)dy, \tag{3.22}$$

$$T_2 f = \int_{-a}^{+a} \Gamma_a(x, y; \lambda)f(y)dy \tag{3.23}$$

are bounded linear transformations from $U[-a, +a]$ to $C[-a, +a]$ or to $U[-a, +a]$.

4. The new functional equation. We next introduce an operator $\tilde{\Omega}_a$ which replaces Ω_a when T_1 and T_2, defined in (3.22) and (3.23), are considered as operators on U instead of on C.

DEFINITION 4.1. *We define*

$$(a^2 - x^2)\tilde{\Omega}_a F = \pi^{-1} P \int_{-a}^{+a} \frac{F'(t)(a^2 - t^2)}{t - x} dt - \pi^{-1} \int_{-a}^{+a} F(t)dt \tag{4.1}$$

for those F in $C[-a, +a]$ which are absolutely continuous in any interval $[x_1, x_2] \subset (-a, +a)$ with $[F']^2(a^2-t^2)^{3/2} \in L[-a, +a]$, and for which the right-hand side is in $C[-a, +a]$.

The new functional equation is then given by

THEOREM 4.1. *If $\Phi = T_1 f$ and $F = T_2 f$ defined by* (3.22) *and* (3.23) *with $f \in U[-a, +a]$, then Φ and F are in the domain of $\tilde{\Omega}_a$; furthermore,*

$$-\tilde{\Omega}_a \Phi = f \tag{4.2}$$

and

$$\lambda F - \tilde{\Omega}_a F = f, \qquad \lambda > 0. \tag{4.3}$$

Also Φ and F are the only solutions in $C[-a, +a]$ of (4.2) *and* (4.3), *respectively.*

PROOF. We first prove (4.2). By Lemma 3.1, we may write Φ' in the form (3.6) with $\phi(y) = f(y)(a^2 - y^2)$. By Theorem 2.2 we see that $[\Phi'(a^2-x^2)^{1/2}]^2(a^2-x^2)^{1/2} \in L[-a, +a]$. Hence, by Theorem 2.1,

$$f(x) = [(a^2 - x^2)\pi]^{-1} \cdot \left\{ \int_{-a}^{+a} f(y)(a^2 - y^2)^{1/2} dy - P \int_{-a}^{+a} \frac{\Phi'(y)(a^2 - y^2)}{y - x} dy \right\}. \tag{4.4}$$

In conclusion we note that

$$\pi^{-1} \int_{-a}^{+a} \Phi(x) dx = \pi^{-1} \int_{-a}^{+a} f(y)(a^2 - y^2)^{1/2} dy, \tag{4.5}$$

using (3.9). Making this substitution in (4.4) we obtain (4.2).

We can now use (4.2) to prove (4.3). From (2.9) we have

$$F(x) + \lambda \int_{-a}^{+a} K_a(x, y) F(y) dy = \Phi(x), \tag{4.6}$$

and the result then follows immediately from the first part of the theorem and the fact that $F \in C[-a, +a]$ which was proved in §3.

To prove the uniqueness, we note that by Theorem 2.1 there is no $\Phi \in C[-a, +a]$ satisfying $\tilde{\Omega}_a \Phi = 0$. If there existed an $F \in C[-a, +a]$ such that $\lambda F - \tilde{\Omega}_a F = 0$, then, by the preceding statement, F would also satisfy (2.7) with $f(x) \equiv 0$ and $A = B = 0$. However, since $K_a(x, y)$ is positive definite, this is impossible for $\lambda > 0$. The proof is now complete.

If F satisfies a certain regularity condition, then $\tilde{\Omega}_a F$ can be written in such a way that its relation to the old operator Ω_a can be seen.

THEOREM 4.2. *If F is absolutely continuous on $(-a, +a)$, then*

$$\tilde{\Omega}_a F(x) = \pi^{-1} P \int_{-a}^{+a} \frac{F'(t)}{t - x} dt - \pi^{-1}[F(a)(a - x)^{-1} + F(-a)(a + x)^{-1}], \tag{4.7}$$

a.e. on $[-a, +a]$.

PROOF. We may write

$$P \int_{-a}^{+a} \frac{F'(t)(a^2 - t^2)}{t - x} dt = (a^2 - x^2) P \int_{-a}^{+a} \frac{F'(t)}{t - x} dt - \int_{-a}^{+a} F'(t)(x + t) dt, \quad \text{a.e.} \tag{4.8}$$

since in this case the first integral on the right exists almost everywhere. An integration by parts in the second integral on the right completes the proof.

It is often convenient to have a sufficient condition on f for the

absolute continuity in $[-a, +a]$ of the functions Φ and F defined in Theorem 4.1. In this case the formulas (4.2) and (4.3) can be written with $\tilde{\Omega}_a$ in the form given by (4.7). To this end we prove

THEOREM 4.3. *If* $f(x)=\phi(x)(a^2-x^2)^{-1}$ *where* $\phi\in C[-a, +a]$ *and*

$$|\phi(x) - \phi(\pm a)| \leqq M(a \pm x)^\alpha \tag{4.9}$$

for some $\alpha>0$ *in some neighborhood* $0<a\pm x<h$, *then* $\Phi=T_1f$ *and* $F=T_2f$, *where* T_i *are defined in* (3.22), (3.23), *are absolutely continuous on* $[-a, +a]$.

PROOF. From Lemma 2.1 we have

$$\pi^{-1}P\int_{-a}^{+a}\frac{(a-y)^{1/2}(a+y)^{-1/2}}{y-x}\,dy = -1 \tag{4.10}$$

and

$$\pi^{-1}P\int_{-a}^{+a}\frac{(a-y)^{-1/2}(a+y)^{1/2}}{y-x}\,dy = 1. \tag{4.11}$$

It will be sufficient to prove the theorem for $\Phi(x)$, by the usual technique of employing (2.9) and the continuity of Φ, since it was shown in [2] that if $\Phi\in C[-a, +a]$, then $T_1\Phi$ defines a function which is absolutely continuous on $[-a, +a]$. Let

$$\phi_1(x) = [\phi(x) - \phi(a)](a-x)^{1/2}(a+x)^{-1/2} \tag{4.12}$$

and

$$\phi_2(x) = [\phi(x) - \phi(a)](a-x)^{-1/2}(a+x)^{1/2}. \tag{4.13}$$

By Lemma 3.1, (4.10) and (4.11) we may write Φ' in the form

$$\Phi'(x) = \pi^{-1}(a^2-x^2)^{-1/2}P\int_{-a}^{+a}\frac{\phi_1(y)+\phi_2(y)}{y-x}\,dy + (a^2-x^2)^{-1/2}[\phi(-a)+\phi(a)]. \tag{4.14}$$

Now for $i=1, 2$, $\phi_i\in L_p(-a, +a)$ for some $p>2$. By Theorem 2.2 the integral in (4.14) is also in $L_p[-a, +a]$ for this choice of p. Hence $F'\in L[-a, +a]$ by Hölder's inequality, and the absolute continuity follows from Lemma 3.1.

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