TWO NOTES ON FORMAL POWER SERIES
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This paper consists of two more or less disjoint notes, the first on integral formal power series in several variables and the second concerning the generalized Puiseux expansion of a certain algebraic function of one variable over a modular field.

1. Analytically independent formal (integral) power series. Let \( k \) be an arbitrary field and let \( L_n \) be the ring of formal series \( k[[x_1, x_2, \ldots, x_n]] \) in \( n \) variables \( x_1, x_2, \ldots, x_n \) with coefficients in \( k \). We recall that given \( m \) elements \( f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n) \) of \( L_n \) one says that \( f_1, f_2, \ldots, f_m \) are analytically dependent if there exists \( 0 \neq H(Z_1, Z_2, \ldots, Z_m) \in k[[Z_1, Z_2, \ldots, Z_m]] \) such that \( H(f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n)) = 0 \). If \( f_1, f_2, \ldots, f_m \) are not analytically dependent then they are said to be analytically independent. Given an infinite number \( f_1, f_2, \ldots \) of elements of \( L_n \), these elements are said to be analytically independent if every finite number of them are analytically independent. Professor Samuel asked me whether \( L_2 \) contains three analytically independent elements. The answer is given in the following

**Proposition.** If \( n > 1 \) then \( L_n \) contains an infinite number of analytically independent elements.

**Proof.** It is known that there exists an infinite number \( g_1(y), g_2(y), \ldots \) of formal power series in one variable \( y \) with coefficients in \( k \) which are algebraically independent over \( k \) (Lemma 1 of [4]). Let \( f_i(x_1, x_2, \ldots, x_n) = x_i g_i(x_1) \) for \( i = 1, 2, \ldots \). Let \( H(Z_1, Z_2, \ldots, Z_m) \) be an element of \( k[[Z_1, Z_2, \ldots, Z_m]] \) for which \( H(f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n)) = 0 \). Let \( H(Z_1, Z_2, \ldots, Z_m) = \sum_{j=0}^{\infty} H_j(Z_1, Z_2, \ldots, Z_m) \) where \( H_j(Z_1, Z_2, \ldots, Z_m) \) is a form of degree \( j \) in \( k[Z_1, Z_2, \ldots, Z_m] \). Then

\[
0 = H(f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n))
\]

\[
= \sum_{j=0}^{\infty} x_1^j H_j(g_1(x_1), g_2(x_1), \ldots, g_m(x_1)).
\]

Therefore

Received by the editors August 23, 1955.
\[ H_i(g_1(x_1), g_2(x_1), \ldots, g_m(x_1)) = \text{the coefficient of } x_i^j \text{ in } H(f_1, \ldots, f_m) \]
\[ = 0 \quad \text{for} \quad j = 1, 2, \ldots. \]

Since \( g_1(x_1), g_2(x_1), \ldots, g_m(x_1) \) are algebraically independent over \( k \), we must have \( H_j(Z_1, Z_2, \ldots, Z_m) = 0 \) for \( j = 1, 2, \ldots \), i.e., \( H(Z_1, Z_2, \ldots, Z_m) = 0 \). Therefore \( f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots \) are analytically independent.

Let us remark that any two elements \( f(x_1) \) and \( g(x_1) \) of \( L_1 \) are analytically dependent. If either \( f \) or \( g \) is zero then there is nothing to prove and hence we may suppose that \( f(x_1) \neq 0 \neq g(x_1) \). First assume that at least one of the two elements \( f(x_1) \) and \( g(x_1) \) is a non-unit; say \( f(x_1) \) is a non-unit. Then by Proposition 3.5 of Chevalley [2], \( L_1 \) is a finite module over \( k[[f(x_1)]] \) and hence \( g(x_1) \) is integral over \( k[[f(x_1)]] \). Therefore there exists \( 0 \neq H(X, Y) \in k[[X]][Y] \) with \( H(f(x_1), g(x_1)) = 0 \). Now assume that \( f(x_1) \) and \( g(x_1) \) are both units. Then \( f^*(x_1) = f(x_1) - ag(x_1) \) is a non-unit, where \( a = f(0)/g(0) \). Hence by the previous case, there exists \( 0 \neq H(X, Y) \in k[[X]][Y] \) with \( H(f^*(x_1), g(x_1)) = 0 \). Let \( H^*(X, Y) = H(X - aY, Y) \). Then \( H^*(f(x_1), g(x_1)) = 0 \). Also \( H^*(X, Y) \neq 0 \) since \( X \to X - aY, Y \to Y \) is an automorphism of \( L_1 \).

2. A fractional power series. Let \( k \) be an algebraically closed field of characteristic \( p \). If \( p = 0 \) then the theorem of Puiseux expansion is valid, i.e., any polynomial \( F(Y) = y^n + f_1(X)Y^{n-1} + \cdots + f_n(X) \) with \( f_i(X) \in k(X) \) can be factored in the form
\[
F(Y) = \prod_{i=1}^n (Y - g_i(X^{1/m})),
\]
where \( m \) is some positive integer and where \( g_1(X), g_2(X), \ldots, g_n(X) \) are in the quotient field of \( k[[X]] \), i.e., in the integral formal power series field \( k((X)) \). If \( p \neq 0 \) then such a factorization is not always possible. A typical example of this is given by Chevalley on p. 64 of [3], namely:
\[
F(Z) = Z^p - Z - X^{-1}.
\]
If we force a factorization, we get the following generalized Puiseux expansion (where the denominators of the indices of \( X \) are unbounded): \[
F(Z) = \prod_{i=0}^{p-1} \left( Z + i - \sum_{i=1}^{\infty} X^{-1/p} \right).
\]
Or getting rid of the poles, we get alternatively:
These factorizations can be verified directly. They were used in discovering some of the examples discussed in [1].

Bibliography


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ON THE COMPOSITUM OF ALGEBRAICALLY CLOSED SUBFIELDS

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Professor Igusa asked me the following question: Given a field $K$, is the compositum of all the (absolutely) algebraically closed subfields of $K$ itself algebraically closed (of course, we are assuming that this compositum is not empty, i.e., that $K$ contains the algebraic closure of its prime field)? We shall show in §1 that the answer to this question is in the negative in general. In §2, we shall give a special case in which the answer is in the affirmative.

1. The algebraic closures of $k(x)$, $k(y)$ and $k(x, y)$. Let $k$ be an arbitrary algebraically closed field, $L = k(x, y)$ where $x$ and $y$ are algebraically independent over $k$, $L^* = $ an algebraic closure of $L$, $M = k(x)$, $N = k(y)$, $M^* = $ the algebraic closure of $M$ in $L^*$, $N^* = $ the algebraic closure of $N$ in $L^*$, and $K = $ the compositum of $L^*$ and $M^*$. Let $T$ be the compositum of all the algebraically closed subfields of $K$. Then $M^* \subset T$ and $N^* \subset T$ and hence $T = K$. We shall prove below that $K$ cannot be algebraically closed. In fact we shall show that in some sense $K$ is much nearer to $L$ than it is to $L^*$ and hence that $K$ is far from being algebraically closed.

Embed $L = k(x, y)$ canonically into the formal power series field.

Received by the editors August 23, 1955.