We establish here the monotonic character of the zeros (modulo 1) of
\[ \int_{z}^{\infty} \frac{f(t)}{t} \, dt, \quad x > 0, \]
where \( f(t) \) satisfies the conditions

(C1). \( f(t) \geq 0 \) for \( 0 \leq t < 1 \);
(C2). \( f(t) \neq 0 \) on any subinterval of \( 0 \leq t < 1 \);
(C3). \( f(t+n) = (-1)^n f(t) \) for \( n = 1, 2, 3, \ldots \);
(C4). \( f(t)/t \) is Lebesgue integrable on \( 0 \leq t \leq 1 \).

It is clear that these conditions imply that the integral (1) has precisely one zero, say \( z_n \), in the interval \( n < x < n+1 \).

Let \( C \) be defined (uniquely) by the conditions

\[ 2 \int_{0}^{C} f(t) \, dt = \int_{0}^{1} f(t) \, dt, \quad 0 < C < 1. \]

Now,

(A) \( z_n - n \geq C \) for \( n = 0, 1, 2, \ldots \),
(B) \( z_n - n \to C \) as \( n \to \infty \),
as was shown in [2], even more generally, with the factor \( 1/t \) of \( f(t) \) in (1) replaced by a function denoted there by \( g(t) \) of which \( 1/t \) is a special case. When \( f(t) = \sin \pi t \), the sequence \( \{ z_n - n \} \) is decreasing, as Harry Pollard has shown, and I. I. Hirschman has observed that Pollard's proof applies equally well to the zeros of

\[ \int_{z}^{\infty} g(t) \sin \pi t \, dt \]
where $g(t)$ is completely monotonic in $0 < t < \infty$ [3, pp. 409–411]. (Here $g(t)$ has a meaning different from the one in [2].)

We prove here the following result:

Let $f(t)$ satisfy the conditions (C1)–(C4), and denote by $z_n$ the unique zero of (1) in the interval $n < x < n + 1, n = 0, 1, 2, \ldots$. Then $z_n - n \downarrow C$, where $C$ is defined by (2).

In replacing $\sin \pi t$ in $\sin(\pi x)$ by a more general function the above result extends Theorem 3.2 of [3] in one direction, while Hirschman's observation concerning (3) generalizes that theorem in another fashion by replacing $1/t$ in $\sin(\pi x)$ by an arbitrary completely monotonic function $g(t)$.

In view of (B), it is only the monotonicity of the sequence $\{z_n - n\}_0^\infty$ that need be established. The formula

$$\int_{z_n}^{1+z_n} f(t)G(t)dt = 0. \tag{5}$$

Suppose that $G_0(t)$ is a non-negative increasing function of $t$ for $0 < t < \infty$. By the second mean-value theorem, (A), and (5),

$$( -1)^n \int_{z_n}^{1+z_n} f(t)G(t)G_0(t)dt = ( -1)^n G_0(1 + z_n) \int_{e_n}^{1+z_n} f(t)G(t)dt \leq 0,$$

where $z_n < x_n < z_{n+1}$. Thus, if there is an $\alpha, n + C \leq \alpha < n + 1$, for which

$$\int_{\alpha}^{1+z_n} f(t)G(t)G_0(t)dt = 0,$$

then, necessarily, $\alpha \leq z_n$. Since

$$0 = \int_{z_n + 1}^{1+z_n+1} f(t)G(t)dt = - \int_{-1+z_n+1}^{z_n+1} f(t)G(t + 1)dt$$

$$= - \int_{-1+z_n+1}^{z_n+1} f(t)G(t) \frac{G(t + 1)}{G(t)} dt,$$
it follows by this argument that \( z_{n+1} - (n + 1) \leq z_n - n \), provided only that \( G(t+1)/G(t) \) is increasing for \( 0 < t < \infty \). We show now that this is the case.

Recalling the definition of \( G(t) \), it is easy to verify that \( G(t) + G(t+1) = 2/t \) [1, p. 20 (7)]. Thus, we may accomplish our aim by showing that \( tG(t) \) is decreasing for \( t > 0 \).

**First proof.** (This was suggested in conversation with M. Riesz.) We have [1, p. 20(2)]

\[
tG(t) = 2t \int_0^1 \frac{r^{t-1}}{1 + r} \, dr.
\]

Integrating by parts,

\[
tG(t) = 1 + 2 \int_0^1 \frac{r^t}{(1 + r)^2} \, dr,
\]

from which the desired result follows immediately.

**Remarks.** On successive differentiation, (6) shows that \( tG(t) \) is completely monotonic, \( 0 < t < \infty \). This is true also of \( t^\delta G(t) \) for any \( \delta < 1 \), since \( t^\delta G(t) \) can be written as the product of the two completely monotonic functions \( 1/t^{1-\delta}, \delta < 1 \), and \( tG(t) \). [That the product of two completely monotonic functions is also completely monotonic follows at once from the successive differentiation of that product by Leibniz's rule.]

Moreover, the restriction \( \delta \leq 1 \) cannot be removed if the function \( t^\delta G(t) \) is to be completely monotonic, \( 0 < t < \infty \), since \( t^\delta G(t) \) increases rather than decreases (as required for complete monotonicity), at least for some positive interval of values of \( t \), for any \( \delta > 1 \).

To see this, let \( \delta = 1 + \epsilon, \epsilon > 0 \). Then

\[
[t^{t+\epsilon}G(t)]' = t'[\{(1 + \epsilon)G(t) + tG'(t)\}]
\]

\[
= 2t^\epsilon \sum_{n=0}^{\infty} \frac{(-1)^n}{t + n} \left(1 + \epsilon - \frac{t}{t + n}\right),
\]

where the infinite series representation is found directly or is taken from [1, p. 20 (6), p. 45 (10)].

This series is an alternating series whose first term is positive. The series itself will be shown to be positive for certain values of \( t \), and the function \( t^{t+\epsilon}G(t) \) to be increasing there, once we show that the terms of that series are monotonically decreasing for those values of \( t \). Now, we observe that this is the case if
We note that the expression in braces decreases as \( n \) increases. Thus, the left member of (7) is greatest when \( n = 0 \), i.e., its maximum is \((2t+1)/(t+1)\). But this maximum is \( \leq 1+\epsilon \) when \( t^{-1} \geq (1-\epsilon)/\epsilon \) and so we have shown that \( t^{1+\epsilon}G(t) \) cannot be completely monotonic, \( 0 < t < \infty \), for any \( \epsilon > 0 \) whatever.

Some interest may attach to the above observation, since \( G(t) \) is a standard "special function" and can be defined in terms of \( \psi(t) \), the logarithmic derivative of the gamma function. Doing so, we can express these results as follows:

The function \( t^\delta [\psi(t+1/2) - \psi(t)] \) is completely monotonic, \( 0 < t < \infty \), if and only if \( \delta \leq 1 \).

If \( 1 < \delta < 2 \), the function increases for \( 0 < t \leq (\delta - 1)/(2 - \delta) \).

If \( \delta \geq 2 \), it increases for all \( t > 0 \).

In case \( \delta = 1 \), this shows [1, p. 20 (6)] that the hypergeometric function \(_2F_1(1, t; 1+t; -1)\) is also completely monotonic, \( 0 < t < \infty \).

SECOND PROOF. To show alternatively that \( tG(t) \) decreases as \( t \) increases, \( 0 < t < \infty \), we put \( 2t = 1/s \) and use [1, p. 20 (6)], whence

\[
2tG(2t) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{1/s + n} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{1 + ns} = 2 \int_{0}^{1} \frac{dr}{1 + r^s}.
\]

The last expression clearly decreases from 2 to 1 as \( s \) decreases from \( \infty \) to 0, that is, as \( t \) increases from 0 to \( \infty \).

REFERENCES

1. A. Erdélyi et al, Higher transcendental functions (based, in part, on notes left by Harry Bateman), vol. 1, New York, 1953.


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