ATOMIC LATTICES WITH UNIQUE COMPARABLE COMPLEMENTS

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1. **Introduction.** It follows immediately from the definitions that if a complemented lattice is modular then no element in the lattice can have distinct comparable complements. For lattices of finite dimension Dilworth [2] has demonstrated a converse . . . *every complemented, nonmodular lattice of finite dimension contains a complemented nonmodular sublattice of order five.* It is the purpose of this note to extend Dilworth's result to atomic lattices; the theorem proved may be stated as follows:

Every complemented, atomic lattice with unique comparable complements is modular.

An easy corollary of this is that *every atomic lattice with unique complements is a Boolean algebra.* This corollary improves the theorem of Birkhoff-Ward [1] stating that every complete atomic lattice with unique complements is a Boolean algebra.

It should be pointed out that the theorem of Dilworth [3] to the effect that every lattice is a sublattice of a lattice with unique complements shows that the word atomic cannot be deleted from the theorem stated above.

Finally, I would like to thank Professor Dilworth for raising the question settled here.

2. **Notation and terminology.** Lattice inclusion will be denoted by $a \supseteq b$, proper inclusion by $a \supset b$, covering by $a > b$. A lattice is said to be *atomic* if it has a null element, z, and if every non-null element contains an element covering z. An element a' is said to be a *complement* of a if $a \cap a' = z$, $a \cup a' = u$, the unit element of the lattice. If every element of the lattice has a complement, the lattice is said to be *complemented.* For $a \supseteq b$ in a lattice the symbol a/b denotes the sublattice of all x with $a \supseteq x \supseteq b$.

3. **Proof of the theorem.** Throughout this section the lattice L to which we refer is a complemented, atomic lattice in which $x \cup y = x \cup w = u$, $x \cap y = x \cap w = z$, $y \supseteq w$ implies $y = w$, that is, comparable complements are unique. We tacitly assume L has at least two elements. The proof that L is modular is made by first showing that the

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Dedekind transposition principle holds for one-dimensional quotients $\cdots a > a \cap b$ *if and only if* $a \cup b > b$. This, together with the uniqueness of comparable complements, is then used to show that L enjoys one of the essential properties of a projective geometry . . . Lemma 5. This in turn is used to show that comparable relative complements are unique and hence that L is modular. I would like to thank the referee for pointing out a superfluous lemma in the original proof of the theorem.

LEMMA 1. *If* $p > z$ *and* $s = p \cup x \supset x$, *then* $u > x \cup t$ *for any complement,* t, *of* s.

It is sufficient to show that $x \cup t$ is a complement of p, since comparable complements are unique. For this one observes that $p \cup (x \cup t) = (p \cup x) \cup t = s \cup t = u$; therefore $p \cup (x \cup t) \supset x \cup t$, again since comparable complements are unique, and hence $p \cap (x \cup t) = z$.

COROLLARY. *For each* $x \neq u$ *in* L *there exists an* m *such that* $u > m \supseteq x$.

LEMMA 2. *If* $p > x \cap p = z$, *then* $p \cup x > x$.

Let $a = p \cup x$ and let b be a complement of a. Then by Lemma 1, $u > x \cup b$. We first show that if $s \neq a$ is in a/x then $t = a \cap (x \cup b) \supseteq s$. This shows in particular that $a > t$. Since $s \supseteq a$, $p \cup s = a$ and therefore $u > s \cup b$. But $s \cup b \supseteq x \cup b$ and $u > x \cup b$ implies $x \cup b = s \cup b$ implies $t \supseteq s$. Now choose a complement, w, of t. To prove the lemma it is sufficient to show that w is also a complement of x. If $x \cup (a \cap w) = a$ we are through, for then $x \cup w \supseteq a$ implies $x \cup w \supseteq a \cup w \supseteq t \cup w = u$. However, if $x \cup (a \cap w) \neq a$ then by our previous observation $t \supseteq x \cup (a \cap w)$ implies $t \supseteq a \cap w$ implies $a \cap w = z$. This supplies a contradiction in the form of w having distinct comparable complements a and t.

LEMMA 3. *If* $x \supset y$, *then there exists* $p > z$, *such that* $x \supseteq p$, $y \cap p = z$.

Suppose $y \supseteq p$ whenever $x \supseteq p > z$ and choose a complement, v, of $x \cap y$. Then $x \cap v \neq z$ and p exists with $x \cap v \supseteq p > z$. Now $y \supseteq p$, a contradiction. This lemma says that any x in L is the union of all the points it contains . . . *whether or not* L *is complete.*

LEMMA 4. *The rule* $a > a \cap b$ *if and only if* $a \cup b > b$ *is valid in* L.

If $a > a \cap b$, then by Lemma 3 there exists $p > z$ such that $a = p \cup (a \cap b)$. Then $a \cup b = p \cup (a \cap b) \cup b = p \cup b > b$ by Lemma 2. The corollary to Lemma 1 gives the dual of Lemma 3, and this combined with the dual of Lemma 2 gives the opposite implication.

LEMMA 5. *If $a \neq u$ and b is a complement of a then for every $p > z$ in L there exist r, s in L such that $a \supseteq r > z$, $b \supseteq s > z$ and $r \cup s \supseteq p$.*

We may assume $a \cap p = b \cap p = z$ or there is nothing to prove. Then $p \cup a > a$ and by the duals of Lemmas 3 and 1, $s = b \cap (p \cup a) > z$. Hence $(p \cup s) \cup a = p \cup a > a$ implies $p \cup s > (p \cup s) \cap a = r$ by the previous lemma. Therefore $p \cup r = p \cup s > p$ implies $r > r \cap p = z$ (again using Lemma 5). This proves the lemma.

LEMMA 6. *If $a \supseteq b$, there exists an element c such that $b \cup c = a$ and $b \cap c = z$.*

Choose any complement, d, of b and put $c = a \cap d$. Combining Lemmas 3 and 5 it is easy to see that this c has the desired properties.

THEOREM. *L is modular.*

PROOF. Suppose $e \supseteq f$ are relative complements of b in a/c. Using Lemma 6 and its dual choose a relative complement, x, for c in b/z and a relative complement, v, for a in u/x. Then e and f are both complements of v and so $e = f$. This proves the theorem.

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