1. Introduction. It follows immediately from the definitions that if a complemented lattice is modular then no element in the lattice can have distinct comparable complements. For lattices of finite dimension Dilworth [2] has demonstrated a converse . . . every complemented, nonmodular lattice of finite dimension contains a complemented nonmodular sublattice of order five. It is the purpose of this note to extend Dilworth's result to atomic lattices; the theorem proved may be stated as follows:

Every complemented, atomic lattice with unique comparable complements is modular.

An easy corollary of this is that every atomic lattice with unique complements is a Boolean algebra. This corollary improves the theorem of Birkhoff-Ward [1] stating that every complete atomic lattice with unique complements is a Boolean algebra.

It should be pointed out that the theorem of Dilworth [3] to the effect that every lattice is a sublattice of a lattice with unique complements shows that the word atomic cannot be deleted from the theorem stated above.

Finally, I would like to thank Professor Dilworth for raising the question settled here.

2. Notation and terminology. Lattice inclusion will be denoted by \( a \supseteq b \), proper inclusion by \( a \succ b \), covering by \( a \succsim b \). A lattice is said to be atomic if it has a null element, \( z \), and if every non-null element contains an element covering \( z \). An element \( a' \) is said to be a complement of \( a \) if \( a \cap a' = z \), \( a \cup a' = u \), the unit element of the lattice. If every element of the lattice has a complement, the lattice is said to be complemented. For \( a \supseteq b \) in a lattice the symbol \( a/b \) denotes the sublattice of all \( x \) with \( a \supseteq x \supseteq b \).

3. Proof of the theorem. Throughout this section the lattice \( L \) to which we refer is a complemented, atomic lattice in which \( x \cup y = x \cup w = u \), \( x \cap y = x \cap w = z \), \( y \supseteq w \) implies \( y = w \), that is, comparable complements are unique. We tacitly assume \( L \) has at least two elements. The proof that \( L \) is modular is made by first showing that the
Dedekind transposition principle holds for one-dimensional quotients \( a > a \cap b \) if and only if \( a \cup b > b \). This, together with the uniqueness of comparable complements, is then used to show that \( L \) enjoys one of the essential properties of a projective geometry. . . Lemma 5. This in turn is used to show that comparable relative complements are unique and hence that \( L \) is modular. I would like to thank the referee for pointing out a superfluous lemma in the original proof of the theorem.

**Lemma 1.** If \( p > z \) and \( s = p \cup x \vee t \), then \( u > x \cup t \) for any complement, \( t \), of \( s \).

It is sufficient to show that \( x \cup t \) is a complement of \( p \), since comparable complements are unique. For this one observes that \( p \cup (x \cup t) = (p \cup x) \cup t = s \cup t = u \); therefore \( p \cup (x \cup t) \supseteq x \cup t \), again since comparable complements are unique, and hence \( p \cap (x \cup t) = z \).

**Corollary.** For each \( x \not\equiv u \) in \( L \) there exists an \( m \) such that \( u > m \supseteq x \).

**Lemma 2.** If \( p > x \cap p = z \), then \( p \cup x > x \).

Let \( a = p \cup x \) and let \( b \) be a complement of \( a \). Then by Lemma 1, \( u > x \cup b \). We first show that if \( s \not\equiv a \) is in \( a / x \) then \( t = a \cap (x \cup b) \supseteq s \). This shows in particular that \( a > t \). Since \( s \supseteq a \), \( p \cup s = a \) and therefore \( s \supseteq b \). But \( s \supseteq b \supseteq x \cup b \) and \( u > x \cup b \) implies \( x \cup b = s \cup b \) implies \( t \supseteq s \). Now choose a complement, \( w \), of \( t \). To prove the lemma it is sufficient to show that \( w \) is also a complement of \( x \). If \( x \cup (a \cap w) = a \) we are through, for then \( x \cup w \supseteq a \) implies \( x \cup w \supseteq a \cup w \cup w = u \). However, if \( x \cup (a \cap w) \not\equiv a \) then by our previous observation \( t \supseteq x \cup (a \cap w) \) implies \( t \supseteq a \cap w \) implies \( a \cap w = z \). This supplies a contradiction in the form of \( w \) having distinct comparable complements \( a \) and \( t \).

**Lemma 3.** If \( x \supseteq y \), then there exists \( p > z \), such that \( x \supseteq p \), \( y \cap p = z \).

Suppose \( y \supseteq p \) whenever \( x \supseteq p > z \) and choose a complement, \( v \), of \( x \cap y \). Then \( x \cap w \not\equiv s \) and \( p \) exists with \( x \cap v \supseteq p > z \). Now \( y \supseteq p \), a contradiction. This lemma says that any \( x \) in \( L \) is the union of all the points it contains . . . whether or not \( L \) is complete.

**Lemma 4.** The rule \( a > a \cap b \) if and only if \( a \cup b > b \) is valid in \( L \).

If \( a > a \cap b \), then by Lemma 3 there exists \( p > z \) such that \( a = p \cup (a \cap b) \). Then \( a \cup b = p \cup (a \cap b) \cup b = p \cup b > b \) by Lemma 2. The corollary to Lemma 1 gives the dual of Lemma 3, and this combined with the dual of Lemma 2 gives the opposite implication.
Lemma 5. If \( a \not\subseteq u \) and \( b \) is a complement of \( a \) then for every \( p > z \) in \( L \) there exist \( r, s \) in \( L \) such that \( a \supseteq r > z \), \( b \supseteq s > z \) and \( r \cup s \supseteq p \).

We may assume \( a \cap p = b \cap p = z \) or there is nothing to prove. Then \( p \cup a > a \) and by the duals of Lemmas 3 and 1, \( s = b \cap (p \cup a) > z \). Hence \( (p \cup s) \cup a = p \cup a > a \) implies \( p \cup s > (p \cup s) \cap a = r \) by the previous lemma. Therefore \( p \cup r = p \cup s > p \) implies \( r > r \cap p = z \) (again using Lemma 5). This proves the lemma.

Lemma 6. If \( a \supseteq b \), there exists an element \( c \) such that \( b \cup c = a \) and \( b \cap c = z \).

Choose any complement, \( d \), of \( b \) and put \( c = a \cap d \). Combining Lemmas 3 and 5 it is easy to see that this \( c \) has the desired properties.

Theorem. \( L \) is modular.

Proof. Suppose \( e \supseteq f \) are relative complements of \( b \) in \( a / c \). Using Lemma 6 and its dual choose a relative complement, \( x \), for \( c \) in \( b / z \) and a relative complement, \( v \), for \( a \) in \( u / x \). Then \( e \) and \( f \) are both complements of \( v \) and so \( e = f \). This proves the theorem.

Bibliography


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