

4. A. N. Milgram and P. C. Rosenbloom, *Harmonic forms and heat conduction I: closed Riemannian manifolds*, Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951).

5. ———, *Heat conduction on Riemannian manifolds II: heat distribution on complexes and approximation theory*, Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951).

6. L. Nirenberg, *A strong maximum principle for parabolic equations*, Comm. on Pure and Appl. Math. vol. 6 (1953).

7. P. C. Rosenbloom, *Notes on partial differential equations*, University of Minnesota (in preparation).

8. A. Tychonoff, *Sur l'équation de la chaleur de plusieurs variables*, Bulletin de l'Université d'État de Moscou. I. (1938).

9. D. V. Widder, *Positive temperatures on an infinite rod*, Trans. Amer. Math. Soc. vol. 55 (1944).

UNIVERSITY OF MINNESOTA

NOTE ON PLURISUBHARMONIC AND HARTOGS FUNCTIONS

H. J. BREMERMANN

1. In a recent paper [7] the author has disproved the conjectures that every plurisubharmonic function is a Hartogs function and that every plurisubharmonic function possesses a plurisubharmonic continuation into the envelope of holomorphy of its domain of definition. While the disproof of the first conjecture is rather elementary the disproof of the second conjecture in [7] involves more powerful means.

Now H. Grauert observed in a discussion with the author that by using the same counterexample but passing from tube domains to Reinhardt circular domains the disproof of the continuation conjecture can be conducted in a more elementary way. We carry out this argument here. It furnishes at the same time also a disproof of the first conjecture.

2. The definitions of the Hartogs functions and the plurisubharmonic functions can be found in Bochner-Martin [3], Lelong [9], Hitotumatu [8], and Bremermann [5] and mainly [7]. Every upper semi-continuous Hartogs function is a plurisubharmonic function. Bochner-Martin conjectured that conversely every plurisubharmonic function is a Hartogs function (Bochner-Martin [3, p. 145], compare

Received by the editors July 12, 1955.

also Bremermann [7]). A consequence of this conjecture would be that every plurisubharmonic function possesses a plurisubharmonic continuation into the envelope of holomorphy of its domain of definition. (For the notion of "envelope of holomorphy" (Regularitätshülle) see Behnke-Thullen [1]. A plurisubharmonic continuation is a function that coincides with the given function in its domain of definition and that is still plurisubharmonic in the larger domain. Analogously we speak of "Hartogs continuations." These continuations are in general, if they exist, not unique.) Further references to the problem of the conjectures are contained in Bremermann [7], there also the following lemma has been proved:

LEMMA. *Let $\phi(z)$ be a Hartogs function in D , then $\phi(z)$ possesses a Hartogs continuation into the envelope of holomorphy $E(D)$ of D .*

We consider in the following Reinhardt circular domains. These are domains in the (schlicht) space of n complex variables w_1, \dots, w_n for which the transformations

$$w_j^* = w_j e^{i\vartheta_j}, \quad \vartheta_1, \dots, \vartheta_n \text{ independent real parameters,}$$

are automorphisms. (Compare Behnke-Thullen [1].) A Reinhardt circular domain can be represented in the form

$$\{(w_1 \cdots w_n) \mid (\log |w_1|, \dots, \log |w_n|) \in D\},$$

where D is a domain in the space of n real variables $x_j = \log |w_j|$. (We shall not consider Reinhardt domains containing points $w_j = 0$, $j = 1, \dots, n$, therefore we can use the open euclidean space of the x_j .) The envelope of holomorphy of a Reinhardt circular domain is the Reinhardt domain $\{(w_1, \dots, w_n) \mid (\log |w_1|, \dots, \log |w_n|) \in C(D)\}$, where $C(D)$ is the convex envelope of D . (Behnke-Stein [2], Thullen [10].)

3. In order to construct a counterexample to the conjectures we now consider the following domain in the plane of two real variables (x_1, x_2) :

$$\begin{aligned} B_1 &= \{(x_1, x_2) \mid |x_1| < 4 \wedge |x_2| < 4\} \\ &\quad - \{(x_1, x_2) \mid |x_1| < 2 \wedge |x_2| < 2\}. \\ B_2 &= \{(x_1, x_2) \mid 0 < x_1 \leq 2 \wedge |x_2| < 1\}. \\ B &= B_1 \cup B_2. \end{aligned}$$

In other words B_1 is a square of side length 8 out of which a square of side length 4 has been removed, B_2 is a rectangle.

In B we define a convex function $V(x_1, x_2)$ as follows:

$$V \equiv 0 \text{ in } B_1,$$

$$V \equiv (2 - x_1)^3 \text{ in } B_2.$$

The function V is twice continuously differentiable in B . It is

$$\sum_{\mu, \nu=1}^2 \frac{\partial^2 V}{\partial x_\mu \partial x_\nu} dx_\mu dx_\nu = \begin{cases} 0 & \text{in } B_1, \\ 6(2 - x_1) dx_1^2 & \text{in } B_2. \end{cases}$$

Therefore the quadratic form is positive semi-definite in B , hence $V(x_1, x_2)$ is a convex function in B . (Compare Bonnesen-Fenchel [4].) We now consider the Reinhardt circular domain

$$R = \{(w_1, w_2) \mid (\log |w_1|, \log |w_2|) \in B\},$$

and the function $V^*(w_1, w_2) = V(\log |w_1|, \log |w_2|)$, where V is the function defined above. V^* depends only upon the moduli $|w_1|$ and $|w_2|$. We shall show that V^* is a plurisubharmonic function in R . We consider the tube domain

$$T_B = \{(z_1, z_2) \mid (x_1, x_2) \in B \wedge |y_1| < \infty \wedge |y_2| < \infty\},$$

where the real part of z_j is x_j and the imaginary part y_j . We continue $V(x_1, x_2)$ from B into the tube domain by defining $V(z_1, z_2) = V(x_1, x_2)$.

A function W defined in a tube domain that does not depend upon the imaginary parts of the variables is plurisubharmonic if and only if its restriction to the base B is convex. This follows from the fact that the Hermitian form

$$\sum_{\mu, \nu} \frac{\partial^2 W}{\partial z_\mu \partial \bar{z}_\nu} dz_\mu d\bar{z}_\nu = \sum_{\mu, \nu} \frac{\partial^2 W}{\partial x_\mu \partial x_\nu} dx_\mu dx_\nu$$

is positive semi-definite if and only if the quadratic form

$$\sum_{\mu, \nu} \frac{\partial^2 W}{\partial x_\mu \partial x_\nu} dx_\mu dx_\nu$$

is positive semi-definite as one immediately computes (Bremermann [6].) Therefore $V(z_1, z_2)$ is plurisubharmonic in the tube domain T_B .

The transformation $w_1 = \exp z_1$, $w_2 = \exp z_2$ maps T_B onto R . The mapping is not one-to-one in the large, but it is locally one-to-one and pseudo-conformal. Now a function is plurisubharmonic in a domain if it is plurisubharmonic in the neighborhood of each point. The property to be plurisubharmonic is invariant with respect to one-to-one pseudo-conformal transformations. Therefore

$$V^*(w_1, w_2) = V(\log w_1, \log w_2) = V(\log |w_1|, \log |w_2|)$$

is plurisubharmonic in R .

4. It is easy to see now that V^* cannot possess a plurisubharmonic continuation into the envelope of holomorphy $E(R)$. $E(R)$ is the Reinhardt circular domain

$$\{(w_1, w_2) \mid (\log |w_1|, \log |w_2|) \in C(B)\},$$

where $C(B)$ is the convex envelope of B , that is the square

$$\{(x_1, x_2) \mid |x_1| < 4 \wedge |x_2| < 4\}.$$

By construction V is zero on the boundary of the domain

$$\tilde{B} = \{(x_1, x_2) \mid |x_1| < 3 \wedge |x_2| < 3\}.$$

And therefore V^* is zero on the boundary of the Reinhardt domain

$$\tilde{R} = \{(w_1, w_2) \mid (\log |w_1|, \log |w_2|) \in \tilde{B}\}.$$

If V^* would have a plurisubharmonic continuation into $E(R)$, then V^* would be plurisubharmonic in \tilde{R} , because the closure of $\tilde{R} \subset R$. Because any plurisubharmonic function assumes its maximum on the boundary we should have

$$V^* \leq 0 \text{ in } \tilde{R}.$$

However by definition we have $V^* > 0$ in

$$R^* = \{(w_1, w_2) \mid (\log |w_1|, \log |w_2|) \in B_2\},$$

and because $B_2 \subset \tilde{B}$ we have $R^* \subset \tilde{R}$. Thus we have a contradiction. Any plurisubharmonic continuation of V^* into $E(R)$ would violate the maximum principle. Therefore V^* cannot possess such a continuation. Conjecture 2 is wrong.

5. V^* is at the same time an example of a plurisubharmonic function that is not a Hartogs function. Suppose V^* would be a Hartogs function. Then it would possess according to the lemma a Hartogs continuation into $E(R)$. Every Hartogs function is a plurisubharmonic function, therefore a Hartogs continuation of V^* would be a plurisubharmonic continuation. Such a continuation does not exist, hence V^* cannot be a Hartogs function. Hence the conjecture of Bochner-Martin is wrong.

6. V^* provides also a counter example to Theorem 10 of Lelong [9, p. 331] and to Proposition 17 of Hitotumatu [8]. Lelong states: In order that a function $V(w_1, \dots, w_n)$ that is defined in a Reinhardt circular domain D_0 and that depends only upon the moduli $|w_1| \cdots |w_n|$ is plurisubharmonic in D_0 it is necessary and sufficient that it is the upper envelope of a sequence of functions of the class

L' in D_0 . The class L' of a domain D is defined as the class of functions $c \cdot \log |f(w_1 \cdots w_n)|$, where f is holomorphic in D and c a positive constant.

Now the functions $c \cdot \log |f|$ are Hartogs functions by definition and the upper envelope of a family of Hartogs functions that is uniformly bounded from above is a Hartogs function by definition. Therefore Theorem 10 would have as a consequence that each such function would be a Hartogs function. The function V^* provides a counterexample.

BIBLIOGRAPHY

1. H. Behnke and P. Thullen, *Theorie der Funktionen mehrerer komplexer Veränderlichen*, Ergebnisse der Mathematik vol. 3, no. 3, 1934.
2. H. Behnke and K. Stein, *Die Konvexität in der Funktionentheorie mehrerer komplexer Veränderlichen*, Mitteilungen der Mathematischen Gesellschaft in Hamburg vol. 8 (1940) pp. 34–81.
3. S. Bochner and W. T. Martin, *Several complex variables*, Princeton, 1948.
4. T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Ergebnisse der Mathematik vol. 3, no. 1, 1934.
5. H. J. Bremermann, *Die Charakterisierung von Regularitätsgebieten durch pseudokonvexe Funktionen*, Thesis, Münster, Schriftenreihe d. Math. Inst. Münster no. 5, 1951.
6. ———, *Complex convexity*, Trans. Amer. Math. Soc. vol. 82 (1956) pp. 17–51.
7. ———, *On the conjecture of the equivalence of the plurisubharmonic functions and the Hartogs functions*, Math. Ann. vol. 131 (1956) pp. 76–86.
8. S. Hitotumatu, *On some conjectures concerning pseudo-convex domains*, J. Math. Soc. Japan vol. 6 (1954) pp. 178–195.
9. P. Lelong, *Les fonctions plurisousharmoniques*, Ann. École Norm. vol. 62 (1945) pp. 301–338.
10. P. Thullen, *De la teoría de las funciones analíticas de varias variables complejas. Dominios de regularidad y dominios de meromorfía de Reinhardt*, Revista de la Unión Matemática Argentina vol. 9 (1945).

Further references are contained in the bibliography of Bremermann [7].

INSTITUTE FOR ADVANCED STUDY