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UNIVERSITY OF SAGAR, INDIA

A NOTE ON HAUSDORFF TRANSFORMS

AMNON JAKIMOVSKI

1. **Introduction.** If $\{\mu_p\}$ ($p=0, 1, 2, \dots$) is an arbitrary sequence of numbers, we shall define the double sequence $(\Delta^n \mu_p)$ ($n, p=0, 1, 2, \dots$) by

$$(1.1) \quad \Delta^n \mu_p = \sum_{r=0}^n (-1)^r \binom{n}{r} \cdot \mu_{p+r} \quad (n, p = 0, 1, 2, \dots).$$

The quantities (1.1) are generally referred to as the "differences" of the sequence $\{\mu_p\}$ since they satisfy the recurrence relations

$$(1.2) \quad \begin{aligned} \Delta^0 \mu_p &= \mu_p & (p = 0, 1, 2, \dots), \\ \Delta^n \mu_p &= \Delta^{n-1} \mu_p - \Delta^{n-1} \mu_{p+1} & (n = 1, 2, 3, \dots; p = 0, 1, 2, \dots). \end{aligned}$$

In this paper, we shall say that the sequence $\{t_p\}$ is the Hausdorff transform of the sequence $\{s_p\}$ generated by the sequence $\{\mu_p\}$ in case

$$(1.3) \quad t_p = \sum_{q=0}^p \binom{p}{q} \cdot (\Delta^{p-q} \mu_q) \cdot s_q \quad (p = 0, 1, 2, \dots).$$

A Hausdorff transform generated by a sequence $\{\mu_p\}$ will be called here, for shortness, an (H, μ_p) transform.

Although transformations of the form (1.3) are generally referred to as Hausdorff transformations, it is to be noted that they were actually first investigated to any extent by Hurwitz and Silverman [1],¹ who characterized them as sequence-to-sequence transformations generated by triangular matrices A permutable with the matrix

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¹ Numbers in square brackets refer to references at the end of this note.

M of $(C, 1)$ summability. In his later investigations, Hausdorff [2] in effect characterizes transformations of the form (1.3) as sequence-to-sequence transformations generated by triangular matrices A and having the property given by the following theorem:

THEOREM 1.1. *If $\{\mu_p\}$ and $\{s_p\}$ are any two sequences of numbers, then $\{t_p\}$ is the (H, μ_p) transform of $\{s_p\}$ if and only if the sequences $\{\Delta^n s_0\}$, $\{\Delta^n t_0\}$ satisfy the multiplicative relation*

$$(1.4) \quad \Delta^n t_0 = \mu_n \cdot \Delta^n s_0 \quad (n = 0, 1, 2, \dots).$$

In §2, we prove a theorem which is a generalization of Theorem 1.1 and which includes this latter theorem as a special case.

If the sequence $\{t_p\}$ is a linear transform of the sequence $\{s_p\}$ of the form

$$(1.5) \quad t_p = \sum_{q=0}^p a_{p,q} s_q \quad (p = 0, 1, 2, \dots),$$

then the sequence $\{t_p^*\}$ defined by

$$(1.6) \quad t_p^* = \sum_{q=0}^p a_{p,p-q} s_q \quad (p = 0, 1, 2, \dots)$$

will be called the "reversed transform" of $\{t_p\}$; it is clear that the reversed transform of $\{t_p^*\}$ is $\{t_p\}$. If $\{\tau_p\}$ is any given sequence of positive numbers, and if $\{s_p\}$ is any sequence of numbers, then the Nörlund transform

$$(1.7) \quad t_p = \frac{\tau_p s_0 + \tau_{p-1} s_1 + \dots + \tau_0 s_p}{\tau_0 + \tau_1 + \dots + \tau_p} \quad (p = 0, 1, 2, \dots)$$

and the transform of M. Riesz

$$(1.8) \quad t_p^* = \frac{\tau_0 s_0 + \tau_1 s_1 + \dots + \tau_p s_p}{\tau_0 + \tau_1 + \dots + \tau_p} \quad (p = 0, 1, 2, \dots)$$

are, for example, reversed transforms of each other; however, it is easy to see that, except for the special case

$$\frac{a^p s_0 + a^{p-1} s_1 + \dots + a^0 s_p}{a^0 + a^1 + \dots + a^p} = \frac{(a^{-1})^0 s_0 + (a^{-1})^1 + \dots + (a^{-1})^p s_p}{(a^{-1})^0 + (a^{-1})^1 + \dots + (a^{-1})^p},$$

where $a > 0$, the sequence $\{t_p\}$ defined by (1.7) cannot be expressed as a transform of the sequence $\{s_p\}$ of the form (1.8). In §3, we note that if the sequence $\{t_p\}$ is a Hausdorff transform of the sequence $\{s_p\}$, then the reversed transform of $\{t_p\}$ is also a Hausdorff trans-

form of $\{s_p\}$. In conclusion, a relationship between this result and results of H. L. Garabedian and H. S. Wall [3] is pointed out.

2. **A generalization of Theorem 1.1.** We first prove

LEMMA 2.1. *If $\{a_p\}$ and $\{b_p\}$ are any two sequences, then*

$$(2.1) \quad \sum_{r=0}^p \binom{p}{r} \cdot \Delta^{p-r} a_{r+n+1} \cdot \Delta^{n+1} b_r = \sum_{r=0}^p \binom{p}{r} \cdot \Delta^{p-r} a_{r+n} \cdot \Delta^n b_r \\ - \sum_{r=0}^{p+1} \binom{p+1}{r} \cdot \Delta^{p+1-r} a_{r+n} \cdot \Delta^n b_r \quad (p, n = 0, 1, 2, \dots).$$

PROOF. For $p, n = 0, 1, 2, \dots$ we have that

$$\sum_{r=0}^p \binom{p}{r} \cdot \Delta^{p-r} a_{r+n+1} \cdot \Delta^{n+1} b_r = \sum_{r=0}^p \binom{p}{r} \cdot \Delta^{p-r} a_{r+n+1} \cdot (\Delta^n b_r - \Delta^n b_{r+1}) \\ = \sum_{r=0}^p \binom{p}{r} \cdot \Delta^{p-r} a_{r+n+1} \cdot \Delta^n b_r - \sum_{r=0}^p \binom{p}{r} \cdot \Delta^{p-r} a_{r+n+1} \cdot \Delta^n b_{r+1} \\ = \sum_{r=0}^p \binom{p}{r} \cdot \Delta^{p-r} a_{r+n} \cdot \Delta^n b_r + \sum_{r=0}^p \binom{p}{r} \cdot [\Delta^{p-r} a_{r+n+1} - \Delta^{p-r} a_{r+n}] \cdot \Delta^n b_r \\ - \sum_{r=0}^p \binom{p}{r} \cdot \Delta^{p-r} a_{r+n+1} \cdot \Delta^n b_{r+1} \\ = \sum_{r=0}^p \binom{p}{r} \cdot \Delta^{p-r} a_{r+n} \cdot \Delta^n b_r - \left\{ \sum_{r=0}^p \binom{p}{r} \cdot \Delta^{p+1-r} a_{r+n} \cdot \Delta^n b_r \right. \\ \left. - \sum_{r=1}^{p+1} \binom{p}{r-1} \cdot \Delta^{p+1-r} a_{r+n} \cdot \Delta^n b_r \right\} \\ = \sum_{r=0}^p \binom{p}{r} \cdot \Delta^{p-r} a_{r+n} \cdot \Delta^n b_r - \left\{ \Delta^{p+1} a_n \cdot \Delta^n b_0 \right. \\ \left. + \sum_{r=1}^p \left[\binom{p}{r} + \binom{p}{r-1} \right] \cdot \Delta^{p+1-r} a_{r+n} \cdot \Delta^n b_r + \Delta^0 a_{p+1+n} \cdot \Delta^n b_{p+1} \right\}.$$

Since

$$\binom{p}{r} + \binom{p}{r-1} = \binom{p+1}{r}, \quad (r = 1, 2, \dots, p; p = 1, 2, 3, \dots),$$

it then follows that (2.1) holds.

This completes the proof of Lemma 2.1.

The next lemma follows from Lemma 2.1 by induction on the index n .

LEMMA 2.2. *If the three sequences $\{a_p\}$, $\{b_p\}$ and $\{c_p\}$ satisfy*

$$(2.2) \quad \Delta^n c_p = \sum_{r=0}^p \binom{p}{r} \cdot \Delta^{p-r} a_{r+n} \cdot \Delta^n b_r$$

for some non-negative integral value of n , say n_0 , and for $p = p_0, p_0 + 1, p_0 + 2, \dots$, where p_0 is some fixed non-negative integer, then the above three sequences satisfy (2.2) for all integral values of n not less than n_0 and for $p = p_0, p_0 + 1, p_0 + 2, \dots$.

We are now ready to prove the following theorem which contains the main result of this section.

THEOREM 2.1. *The sequences $\{\mu_p\}$, $\{s_p\}$ and $\{t_p\}$ satisfy*

$$(2.3) \quad t_p = \sum_{q=0}^p \binom{p}{q} \cdot \Delta^{p-q} \mu_q \cdot s_q \quad (p = p_0, p_0 + 1, p_0 + 2, \dots),$$

where p_0 is some fixed non-negative integer, if and only if

$$(2.4) \quad \Delta^n t_{p_0} = \sum_{q=0}^{p_0} \binom{p_0}{q} \cdot \Delta^{p_0-q} \mu_{q+n} \cdot \Delta^n s_q \quad (n = 0, 1, 2, \dots).$$

PROOF. The necessity of (2.4) in order that the sequences $\{\mu_p\}$, $\{s_p\}$ and $\{t_p\}$ satisfy (2.3) is a consequence of Lemma 2.2.

Suppose, on the other hand, that the sequences $\{\mu_p\}$, $\{s_p\}$ and $\{t_p\}$ satisfy (2.4). Let $\{t'_p\}$ be any sequence such that

$$t'_p = \sum_{q=0}^p \binom{p}{q} \cdot \Delta^{p-q} \mu_q \cdot s_q \quad (p = p_0, p_0 + 1, p_0 + 2, \dots).$$

From the preceding paragraph it follows that

$$\Delta^n t'_{p_0} = \sum_{q=0}^{p_0} \binom{p_0}{q} \cdot \Delta^{p_0-q} \mu_{q+n} \cdot \Delta^n s_q \quad (n = 0, 1, 2, \dots).$$

Thus, we have that

$$\Delta^n t'_{p_0} = \Delta^n t_{p_0} \quad (n = 0, 1, 2, \dots).$$

It follows by induction, then, that

$$t'_{p_0+n} = t_{p_0+n} \quad (n = 0, 1, 2, \dots).$$

Therefore

$$t_p = \sum_{q=0}^p \binom{p}{q} \cdot \Delta^{p-q} \mu_q \cdot s_q \quad (p = p_0, p_0 + 1, p_0 + 2, \dots).$$

This completes the proof of Theorem 2.1.

3. **The reversed transform of a Hausdorff transform.** Hausdorff [2] has proved the following result.

LEMMA 3.1. *If the sequences $\{\mu_p\}$ and $\{\lambda_p\}$ satisfy*

$$(3.1) \quad \lambda_p = \Delta^p \mu_0 \quad (p = 0, 1, 2, \dots),$$

then

$$(3.2) \quad \Delta^n \lambda_p = \Delta^p \mu_n \quad (n, p = 0, 1, 2, \dots).$$

From Lemma 3.1 and the equality

$$\binom{p}{p-q} = \binom{p}{q} \quad (q = 0, 1, 2, \dots, p; p = 0, 1, 2, \dots).$$

we have at once the following theorem which contains the main result of this section.

THEOREM 3.1. *If the sequence $\{t_p\}$ is the (H, μ_p) transform of the sequence $\{s_p\}$, then the reversed transform of $\{t_p\}$ is the (H, λ_p) transform of $\{s_p\}$, where the sequence $\{\lambda_p\}$ satisfies (3.1).*

In case $\{\mu_p\}$ is a monotone Hausdorff moment sequence, then in view of Theorem 3.1 we have from Theorem 3.1 and Theorem 3.3 given by H. L. Garabedian and H. S. Wall [3] conditions in terms of the S -fraction expansion of the formal power series $\sum_{p=0}^{\infty} \mu_p w^{p+1}$ for the reversed transformation of the sequence-to-sequence transformation (1.3) to be regular.

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