PARTITIONS OF MULTI-PARTITE NUMBERS

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1. Introduction. In what follows all small latin letters denote non-negative rational integers. We suppose for the present that $|X_i| < 1$ $(1 \leq i \leq j)$ and write

$$F_i(Y) = F_i(X_1, \cdots, X_i; Y) = \prod (1 + X_i^{k_1} \cdots X_i^{k_j} Y)$$

and

$$G_i(Y) = \{F_i(-Y)\}^{-1} = \prod (1 - X_i^{k_1} \cdots X_i^{k_j} Y)^{-1},$$

where the products extend over all non-negative $k_1, \cdots, k_j$. If $|Y| < 1$, we have

$$G_i(Y) = 1 + \sum_{n=1}^{\infty} Q_i(n) Y^n,$$

where

$$Q_i(n) = Q_i(X_1, \cdots, X_i; n) = \sum_{n_1, \cdots, n_j=0}^{\infty} q(n_1, \cdots, n_j; n) X_1^{n_1} \cdots X_i^{n_i}$$

and $q(n_1, \cdots, n_j; n)$ is the number of partitions of the $j$-partite number $(n_1, \cdots, n_j)$ into just $n$ parts, that is, the number of solutions of the "vector" equation (or equation in single row matrices)

$$\sum_{k=1}^{n} (x_{1k}, \cdots, x_{jk}) = (n_1, \cdots, n_j).$$

The order of the vectors on the left-hand side of (1) is irrelevant. Again

$$F_i(Y) = 1 + \sum_{n=1}^{\infty} R_i(n) Y^n,$$

where

$$R_i(n) = \sum r(n_1, \cdots, n_j; n) X_1^{n_1} \cdots X_i^{n_i}$$

and $r(n_1, \cdots, n_j; n)$ is the number of partitions of $(n_1, \cdots, n_j)$ into just $n$ different parts, that is, the number of solutions of (1) in which the vectors on the left hand side are all different.

Received by the editors July 13, 1955.
If \( j=1 \), we have

\[(1 - Y)G_1(Y) = G_1(X_1Y)\]

and so

\[Q_1(n) - Q_1(n - 1) = X_1^n Q_1(n),\]

whence

\[Q_1(n) = \frac{Q_1(n - 1)}{1 - X_1^n} = \frac{1}{(1 - X_1)(1 - X_1^2) \cdots (1 - X_1^n)}.\]

Similarly we find that

\[R_1(n) = \frac{X_1^{n(n-1)/2}}{(1 - X_1)(1 - X_1^2) \cdots (1 - X_1^n)}.\]

Macmahon (Combinatory analysis ii, Cambridge, 1916) discussed in detail the case \( j=1 \) and referred briefly to the more general case, commenting on its complexity. More recently Bellman (Bull. Amer. Math. Soc. Research Problem 61-1-3) has asked for a formula for \( Q_2(n) \). My object here is to obtain formulae for \( Q_j(n) \) and \( R_j(n) \) for general \( j \) and \( n \). For \( j>1 \), these formulae cannot be reduced to anything as simple as in the case \( j=1 \), but we can make some progress in this direction and deduce certain results about partitions.

2. The formulae for \( Q_j(n) \) and \( R_j(n) \). Let

\[\alpha_1, \alpha_2, \alpha_3, \cdots\]

be any infinite sequence such that \(|\alpha_k| < 1\) for every \( k \) and \( \sum |\alpha_k| < \infty \). We write

\[C(Y) = \prod_{k=1}^{\infty} (1 + \alpha_k Y) = 1 + \sum_{n=1}^{\infty} A(n)Y^n\]

and

\[D(Y) = \{C(-Y)\}^{-1} = \prod_{k=1}^{\infty} (1 + \alpha_k Y + \alpha_k^2 Y^2 + \cdots) = 1 + \sum_{n=1}^{\infty} B(n)Y^n.\]

Clearly \( A(n) \) is the sum of the products of every set of \( n \) different \( \alpha \) and \( B(n) \) is the sum of the products of every set of \( n \) numbers \( \alpha \).
repetitions permitted. We write also
\[ S(m) = \sum_k a_k^m. \]

We see at once that
\[ \log D(Y) = -\sum_{k=1}^\infty \log (1 - \alpha_k Y) = \sum_{m=1}^\infty \frac{S(m)}{m} Y^m. \]

Hence
\[ D(Y) = \exp \left\{ \sum_{m=1}^\infty \frac{S(m)}{m} Y^m \right\}, \]
and
\[ B(n) = \sum_{(n)} \prod \frac{\{S(m)\}^{h_m}}{h_m! m^{h_m}}, \]
the sum extending over all partitions of \( n \) of the form
\[ n = \sum h_m m \]
and the product over all the different parts \( m \) in the partition. Again
\[ C(-Y) = \exp \left\{ -\sum_{m=1}^\infty \frac{S(m)}{m} Y^m \right\} \]
and so
\[ A(n) = (-1)^n \sum_{(n)} \prod \frac{(-1)^{h_m}\{S(m)\}^{h_m}}{h_m! m^{h_m}}. \]

Next, if we differentiate (3) with respect to \( Y \) and multiply through by \( D(Y) \), we obtain
\[ \sum_{n=1}^\infty n B(n) Y^{n-1} = \sum_{m=1}^\infty S(m) Y^{m-1} \left\{ 1 + \sum_{n=1}^\infty B(n) Y^n \right\} \]
and so, equating coefficients of \( Y^{n-1} \), we have
\[ nB(n) = \sum_{m=1}^n S(m) B(n - m). \]
Similarly
\[ nA(n) = \sum_{m=1}^n (-1)^{m-1} S(m) A(n - m). \]
If we now take all of
\[ X_1^{k_1} \cdots X_j^{k_j} \quad (k_i \geq 0, 1 \leq i \leq j) \]
for the \( \alpha \) in (2), we see that
\[
A(n) = R_j(n), \quad B(n) = Q_j(n)
\]
and
\[
S(m) = \sum X_1^{m_1} \cdots X_j^{m_j} = \prod_{i=1}^{j} \left( \frac{1}{1 - X_i^m} \right) = \frac{1}{\beta_j(m)},
\]
where
\[
\beta_j(m) = \prod_{i=1}^{j} (1 - X_i^m).
\]
Hence we have
\[
Q_j(n) = \sum_{\alpha} \prod (\alpha_m!)^{-1}\{m\beta_j(m)\}^{-h_m}
\]
and
\[
R_j(n) = (-1)^n \sum_{\alpha} \prod (-1)^{h_m}(\alpha_m!)^{-1}\{m\beta_j(m)\}^{-h_m}.
\]
These are the formulae for \( Q_j(n) \) and \( R_j(n) \). For \( j = 1 \), they were found by Macmahon (loc. cit.).

Again, (4) and (5) become
\[
nQ_j(n) = \sum_{m=1}^{n} \frac{Q_i(n-m)}{\beta_j(m)}
\]
and
\[
nR_j(n) = \sum_{m=1}^{n} (-1)^n \frac{R_i(n-m)}{\beta_j(m)}
\]
If \( \sum h_m m = n \), it is easy to show that
\[
\frac{(1 - X)(1 - X^2) \cdots (1 - X^n)}{\prod(1 - X^m)^{h_m}}
\]
is a polynomial in \( X \). Its degree is clearly
\[
\sum_{h_m} k - \sum h_m m = \frac{1}{2} n(n + 1) - n = \frac{1}{2} n(n - 1).
\]
Hence, if we write
we see from (6) that $P_j(n)$ is a polynomial of degree at most $n(n-1)/2$ in each of $X_1, \ldots, X_j$.

It follows from its definition that $Q_j(n)$ is a multiple infinite power series in $X_1, \ldots, X_j$, the coefficient of each term being a non-negative integer. Since the $\beta$ are polynomials with integral coefficients, we see that all the coefficients in the polynomial $P_j(n)$ are integers. It seems very likely that all these coefficients are non-negative, but this I have not been able to prove. In §1, we saw that

$$P_1(n) = 1$$

for all $n$. Unfortunately nothing so simple is true for $j > 1$.

3. Properties of $P_j(n)$. We now suppose that $Q_j(n)$ and $R_j(n)$ are defined by (6) and (7), so that $Q_j(n)$ and $R_j(n)$ are rational functions defined for all values of the $X_i$ except the $m$th roots of unity for which $1 \leq m \leq n$. Again, since $P_j(n)$ is a polynomial, it can be defined for all values of the $X_i$ without exception. We write $P_j(0) = Q_j(0) = R_j(0) = 1$ and see that $P_j(1) = 1.$

We have now

$$\beta_j(X_1, \ldots, X_{i-1}, X_i^{-1}; m) = (1 - X_i^m) \cdots (1 - X_{i-1}^m)(1 - X_i^{-m})$$

$$= - X_i^{-m} \beta_j(X_1, \ldots, X_{i-1}, X_i; m).$$

Hence, by (6) and (7),

$$Q_j(X_1, \ldots, X_{i-1}, X_i^{-1}; n) = X_j^n \sum_{(n)} \prod (-1)^{h_m} (h_m)_{-1} \{m\beta_j(m)\}^{-h_m}$$

and

$$R_j(X_1, \ldots, X_{i-1}, X_i^{-1}; n) = (-1)^n X_j^n Q_j(X_1, \ldots, X_i; n).$$

This transformation applies also with any one of $X_1, \ldots, X_{i-1}$ in place of $X_j$. Applying it twice, we have

$$Q_j(X_1, \ldots, X_{i-2}, X_{i-1}^{-1}, X_i^{-1}; n) = X_{i-1}^n X_i^n Q_j(X_1, \ldots, X_i; n).$$

Using (9), we see that

$$R_j(X_1, \ldots, X_i; n) = \frac{X_i^{n(n-1)/2} P_j(X_1, \ldots, X_{i-1}, X_i^{-1}; n)}{\beta_j(1) \cdots \beta_j(n)}.$$
so that, if we can evaluate $P_j(n)$, we have a simple form for both $Q_j(n)$ and $R_j(n)$. Again

$$(X_{j-1}X_j)^{n(n-1)/2} P_j(X_1, \ldots, X_{j-2}, X_{j-1}^{-1}, X_j^{-1}; n) = P_j(X_1, \ldots, X_j; n).$$

If, then, we write

$$g = n(n-1)/2$$

and

$$P_j(X_1, \ldots, X_j; n) = \sum_{k_1, \ldots, k_j=0}^g \lambda(k_1, \ldots, k_j) X_1^{k_1} \cdots X_j^{k_j}$$

we have

$$\lambda(k_1, \ldots, k_{j-2}, k_{j-1}, k_j) = \lambda(k_1, \ldots, k_{j-2}, g-k_{j-1}, g-k_j)$$

and similarly for any other pair of the $k_i$. We can see at once by putting $X_1 = X_2 = \cdots = X_j = 0$, that $\lambda(0, 0, \ldots, 0) = 1$. Hence $\lambda(g, g, 0, 0, \ldots, 0) = 1$ and so on. It follows that, for $j \geq 2$, $P_j(n)$ is of degree exactly $g$ in every $X_i$.

Next, we see that, in the sum on the right-hand side of (6), the factor $(1-X_j)^n$ occurs in the denominator only in the term in which $m = 1$, $h = n$, i.e. the term corresponding to the partition of $n$ into $n$ units. But the factor $(1-X_j)^n$ occurs in $\beta_j(1)\beta_j(2) \cdots \beta_j(n)$ and so

$$P_j(X_1, \ldots, X_{j-1}, 1; n) = \lim_{x_{j-1} \to 1} \frac{\beta_j(1) \cdots \beta_j(n)}{n!\{\beta_j(1)\}^n}$$

$$= \frac{\beta_j(1) \cdots \beta_j(n)}{\{\beta_j(1)\}^n}$$

$$= \prod_{i=1}^{j-1} \prod_{m=2}^{n} (1 + X_i + X_i^2 + \cdots + X_i^{m-1}).$$

Hence

$$\sum_{k_j=0}^g \lambda(k_1, \ldots, k_j)$$

is the coefficient of $\prod_{i=1}^{j-1} X_i^{k_i}$ in the double product on the right-hand side of (12). Also, putting

$$X_1 = X_2 = \cdots = X_{j-1} = 1,$$

we have

$$P_j(1, 1, \ldots, 1; n) = \sum_{k_1, \ldots, k_j=0}^g \lambda(k_1, \ldots, k_j) = (n!)^{j-1}.$$
Again
\[ P_j(X_1, \ldots, X_{j-1}, 0; n) = P_{j-1}(X_1, \ldots, X_{j-1}; n) \]
and so
\[ \lambda(k_1, \ldots, k_{j-1}, 0) = \lambda(k_1, \ldots, k_{j-1}). \]
By (10), \( \lambda(k_1) = 0 \) unless \( k_1 = 0 \). Hence
\[ \lambda(k_1, 0, 0, \ldots, 0) = 0, \]
unless \( k_1 = 0 \). Thus there is no term in \( P_j \) which consists of a power of one \( X \) only, i.e. apart from the term of zero degree, viz. 1, every term contains at least two of the \( X \). A number of other properties of the \( \lambda \) may be obtained similarly.

From (8) and (9), it follows that
\[
(13) \quad nP_j(n) = \sum_{m=1}^{n} \frac{\beta_j(n - m + 1) \cdots \beta_j(n)}{\beta_j(m)} P_j(n - m).
\]
For \( m \geq 2 \), the factor \( 1 - X_i \) occurs at least once more in the numerator of
\[
\frac{\beta_j(n - m + 1) \cdots \beta_j(n)}{\beta_j(m)}
\]
than in the denominator. Hence
\[ nP_j(n) = \frac{\beta_j(n)}{\beta_j(1)} P_j(n - 1) + \beta_j(1)T, \]
where \( T \) is a polynomial in the \( X_i \).

For a small value of \( m \), we can find the terms containing \( X_i^n \) in \( P_j(n) \) as follows. It is easily verified that
\[ G_j(X_i Y)G_{j-1}(Y) = G_j(Y) \]
and so
\[ \sum_{n=0}^{\infty} Q_j(n) Y^n = \left\{ \sum_{i=0}^{\infty} Q_j(i) X_i Y^i \right\} \left\{ \sum_{s=0}^{\infty} Q_{j-1}(s) Y^s \right\}, \]
whence
\[ (1 - X_i^n)Q_j(n) = \sum_{i=0}^{n-1} X_i^n Q_j(i) Q_{j-1}(n - i), \]
that is
\[ P_j(n) = \frac{1}{\beta_j(1) \beta_j(2)} \left\{ \frac{1}{\{\beta_j(1)\}^2} + \frac{1}{\beta_j(2)} \right\} \]

where, as usual, each empty product denotes unity. The terms in \( X_j \) occur in the first \( m + 1 \) terms on the right and can be expressed in terms of \( P_j(l) \) and \( P_{j-1}(n-l) \) for \( 1 \leq l \leq m \). Thus the term in \( X_j \) is

\[ X_j \left\{ \frac{\beta_{j-1}(n)}{\beta_{j-1}(1)} P_{j-1}(n - 1) - P_{j-1}(n) \right\}. \]

4. Calculation of \( P_j(2) \) and \( P_j(3) \). By (6) and (9),

\[ P_j(2) = \frac{\beta_j(1)\beta_j(2)}{2} \left\{ \frac{1}{\{\beta_j(1)\}^2} + \frac{1}{\beta_j(2)} \right\} \]

\[ = \frac{1}{2} \left( \frac{\beta_j(2)}{\beta_j(1)} + \beta_j(1) \right) \]

\[ = \frac{1}{2} \left\{ \prod_{i=1}^{j} (1 + X_i) + \prod_{i=1}^{j} (1 - X_i) \right\} \]

\[ = 1 + \sum X_1X_2 + \sum X_1X_2X_3X_4 + \cdots. \]

Similarly, since \( 3 = 2+1 = 1+1+1 \), we have

\[ P_j(3) = \frac{\beta_j(1)\beta_j(2)\beta_j(3)}{6} \left\{ \frac{1}{\{\beta_j(1)\}^2} + \frac{1}{2\beta_j(1)\beta_j(2)} + \frac{1}{3\beta_j(3)} \right\} \]

\[ = \frac{1}{6} \left\{ \frac{\beta_j(2)\beta_j(3)}{\{\beta_j(1)\}^2} + 3\beta_j(3) + 2\beta_j(1)\beta_j(2) \right\} \]

\[ = \frac{1}{6} \left\{ \prod_{i=1}^{j} (1 + X_i)(1 + X_i + X_i^2) + 3 \prod_{i=1}^{j} (1 - X_i^2) + 2 \prod (1 - X_i)(1 - X_i^2) \right\}. \]

Now

\[ \prod (1 + 2X_i + 2X_i^2 + X_i^3) + 3 \prod (1 - X_i^3) + 2 \prod (1 - X_i - X_i^2 + X_i^3) \]

\[ = 3 \prod (1 + X_i^3) + 3 \prod (1 - X_i^3) + \sum_{a=2}^{j} \{2^a + (-1)^a2\} \]

\[ \cdot \sum X_1 \cdots X_a(1 + X_1) \cdots (1 + X_a)(1 + X_{a+1}) \cdots (1 + X_j) \]

and so
\[ P_j(3) = 1 + \sum X_1^3 X_2^3 + \sum X_1^3 X_2^2 X_3^2 X_4^2 + \cdots \]
\[ \quad + X_1 \cdots X_j(1 + X_1) \cdots (1 + X_j) \sum_{b=0}^{j-2} \frac{1}{3} \{ 2^{j-b-1} + (-1)^{j-b} \} \]
\[ \quad \cdot \sum \left( X_1 - 1 + \frac{1}{X_1} \right) \cdots \left( X_b - 1 + \frac{1}{X_b} \right). \]

Macmahon (loc. cit.) gives the above form of \( P_j(2) \), but dismisses \( P_j(3) \) with the remark that it is very complex.

From the above, we have
\[ P_2(2) = 1 + X_1 X_2, \quad P_3(2) = 1 + X_1 X_2 + X_2 X_3 + X_3 X_1 \]
and
\[ P_j(3) = 1 + X_1^3 X_2^3 + X_1 X_2 (1 + X_1)(1 + X_2), \]
\[ P_j(3) = 1 + X_1^3 X_2^3 + X_2^3 X_3^3 \]
\[ \quad + X_1 X_2 X_3 (1 + X_1)(1 + X_3)(1 + X_3) \left\{ \sum X_1 - 2 + \sum \frac{1}{X_1} \right\}. \]

The formulae (6) and (9) enable one to evaluate \( P_j(n) \) for small \( j \) and \( n \) and, in particular, to pick out the coefficient of any given term.

5. The case \( j = 2 \). By (12), we see that
\[ P_2(X_1, 1; n) = \prod_{m=2}^{n} \left( 1 + X_1 + X_1^2 + \cdots + X_1^{n-1} \right) = \prod_{m=2}^{n} \left( \frac{1 - X_1^m}{1 - X_1} \right). \]
We see then that
\[ P_2(X_1, X_2; n) - \prod_{m=2}^{n} \left( \frac{1 - X_1^m X_2^m}{1 - X_1 X_2} \right) \]
vanishes when \( X_2 = 1 \) and similarly when \( X_1 = 1 \). It also vanishes in virtue of (10), when \( X_1 = 0 \) and when \( X_2 = 0 \). It follows that
\[ P_2(X_1, X_2; n) = \prod_{m=2}^{n} \left( \frac{1 - X_1^m X_2^m}{1 - X_1 X_2} \right) + X_1 X_2 (1 - X_1)(1 - X_2) M(X_1, X_2; n), \]
where \( M \) is a polynomial in \( X_1 \) and \( X_2 \). Since
\[ X_1^2 X_2^2 P_2(X_1^{-1}, X_2^{-1}; n) = P_2(X_1, X_2; n) \]
and a similar relation is true for the first term on the right-hand side of (15), we must have
\[ X_1^{g-3} X_2^{g-3} M(X_1^{-1}, X_2^{-1}; n) = M(X_1, X_2; n) \]
and so \( M \) is of degree at most \( g - 3 \) in \( X_1 \) and \( X_2 \).

For a fixed \( j \), the recurrence formula (13) provides a slightly less laborious means of finding \( P_j(n) \) than does (6). If we write \( Z = X_1 X_2 \) and
\[ \zeta_0 = 1 + Z + Z^2 + \cdots + Z^m, \]
the values of \( P_2(4) \) and \( P_2(5) \) found from (13) are
\[ P_2(4) = \zeta_4 - Z \zeta_2 \theta(1) - Z \zeta_2 \theta(2) = (1 + Z^2) \zeta_4 + Z \zeta_2 (X_1 + X_2) + Z \zeta_4 (X_1^2 + X_2^2) \]
and
\[ P_2(5) = \zeta_5 - Z \zeta_3 \theta(1) - Z \zeta_3 \theta(2) - Z (1 + Z^2) \zeta_4 (X_1 + X_2) + Z^2 \zeta_4 (X_1^2 + X_2^2) = 1 + Z + 3Z^2 + 6Z^3 + 4Z^4 + 6Z^5 + 4Z^6 + 3Z^7 + 2Z^8 + Z^9 + Z^{10} + Z \zeta_2 (X_1 + X_2) + Z \zeta_4 (X_1^2 + X_2^2) + Z (1 + Z^2) \zeta_4 (X_1 + X_2) + Z^2 \zeta_4 (X_1^2 + X_2^2). \]
The detailed calculations have no point of interest.

6. Consequences in partition-theory. If
\[ \frac{1}{(1 - X)(1 - X^2) \cdots (1 - X^n)} = 1 + \sum_{t=1}^{\infty} p_n(t) X^t, \]
then \( p_n(t) \) is the number of partitions of \( t \) into parts not greater than \( n \). It is well known (see, for example, Hardy and Wright, Theory of numbers, 3d ed., Oxford, 1955, Theorem 343) that \( p_n(t) \) is also the number of partitions of \( t \) into not more than \( n \) parts. From the definition of \( Q_j(n) \) and \( P_j(n) \), we see that
\[ q(n_1, \ldots, n_j; n) = \sum_{k_1, \ldots, k_j=0}^{q} \lambda(k_1, \ldots, k_j) \prod_{i=1}^{j} p_n(n_i - k_i). \]
Hence, if we calculate \( P_j(n) \), we can express \( q(n_1, \ldots, n_j; n) \) in terms of the \( p_n \). Again
\[ r(n_1, \ldots, n_j; n) = \sum_{k_1, \ldots, k_j=0}^{q} \lambda(k_1, k_2, \ldots, k_{j-1}, g - k_j) \prod_{i=1}^{j} p_n(n_i - k_i). \]
7. An asymptotic expansion for large \( n \). For fixed \( X_i \) such that \( |X_i| < 1 \) \((1 \leq i \leq j)\) we can find an asymptotic expansion of \( Q_2(n) \) for large \( n \). For simplicity, we confine ourselves to the case in which \( j = 2, X_1 \) and \( X_2 \) are real and positive and the ratio of their logarithms is not rational, so that \( X_1^m = X_2^n \) is impossible for any positive integral \( u \) and \( v \). In the complex \( Y \)-plane, \( G_2(Y) \) has a simple pole at each of the points

\[
X_1^{-t_1}X_2^{-t_2} \quad (t_1, t_2 \geq 0).
\]

If we write \( \delta = \min \left( |X_1|^{-1}, |X_2|^{-1} \right) \),

\[
\phi(\alpha, X) = \prod_{k=0}^{\infty} (1 - \alpha X^k)^{-1},
\]

\[
J = \phi(X_1, X_1)\phi(X_2, X_2) \prod_{k_1=1}^{\infty} \prod_{k_2=1}^{\infty} (1 - X_1^{k_1}X_2^{k_2})^{-1},
\]

and

\[
K(t_1, t_2; X_1, X_2)
\]

\[
= \prod_{k_1=1}^{t_1} \prod_{k_2=1}^{t_2} (1 - X_1^{-k_1}X_2^{-k_2})^{-1} \prod_{k_2=1}^{t_2} \phi(X_2^{-k_2}, X_1) \prod_{k_1=1}^{t_1} \phi(X_1^{-k_1}, X_2),
\]

we find that

\[
G_2(Y) = J \sum_{h=0}^{m+1} \sum_{k_1=0}^{h} \frac{K(k_1, h - k_1; X_1, X_2)}{1 - X_1^{k_1}X_2^{h-k_1}Y}
\]

is regular on and within the circle \( |Y| = \delta^{m+1} \). It follows that

\[
Q_2(n) = J \sum_{h=0}^{m} \sum_{k_1=0}^{h} K(k_1, h - k_1; X_1, X_2)X_1^{n_1}X_2^{n_2} + O(\delta^{m+1}),
\]

where the \( O(\quad) \) symbol refers to the passage of \( n \) to infinity.

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