A NOTE ON THE ALGEBRA OF BOUNDED FUNCTIONS. II

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1. Let K be a commutative B^*-algebra with identity 1 (with $\|k^*k\| = \|k\|^2$ for all $k \in K$, and $\|1\| = 1$). Then K is equivalent (isomorphic in a norm and * preserving manner) to the algebra $C(M)$ of all continuous complex-valued functions on the compact Hausdorff space M (its structure space) [1; 2]. In [5] we have given necessary and sufficient conditions that K be equivalent to $B(X)$, the ring of all bounded complex-valued functions on the discrete space X. These conditions were ideal-theoretic, involving the annulets (annihilating ideals) of K, and did not depend on the representation $C(M)$. Using the representation $C(M)$ two further characterizations of $B(X)$ are given in [3], one involving the properties of the space M, and the other the notion of projection. The characterizations in [3] are derived independently of the one in [5], and in fact no attempt is made in [3] to relate directly the ideal-theoretic conditions with the notions used there. In this note, we show how the characterizations in [3] can be derived from the characterization in [5] by relating directly the ideal-theoretic properties of K with the properties of the structure space M, and with the idea of projection. In particular, the lattice of annulets of K is anti-isomorphic to the lattice of regular open sets in M (Lemma 1). Another characterization of $B(X)$ (Theorem 4) is a byproduct of our procedure.

2. The notation will follow that in [5]. If $G \leqq K$ then $R(G)$ is the set of all functions $k \in K = C(M)$ such that $kg = 0$ for all $g \in G$. Such ideals were called annulets. $N(G)$ is the set of $y \in M$ such that $g(y) = 0$ for all $g \in G$. If $S \leqq M$, then $A(S)$ is the set of functions f such that $f(x) = 0$ for all $x \in S$. Since $N(G) = \bigcap_{g \in G} N(g)$, it is a closed set. Now by following the arguments of Lemma 1 of [5], and using the fact that if O is open in a compact space and $x \in O$, there exists a function $f \in K$ with $f(x) = 1$ and $f(O') = 0$, we have

(1) $R(G) = A[N(G)']$ for $G \leqq K$;

(2) $R[A(S)] = A(S')$ if S is a closed subset of M;

(3) $N[A(S)] = S$ if S is a closed subset of M.

Now (1) and (2) show that an annulet of K is the set of all functions vanishing on an open set of M and conversely. Since $A(S) = A(\bar{S})$

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and $S \leqq \text{int } \overline{S} \leqq \overline{S}$ for any open set S, an annulet is the set of functions vanishing on a regular open set and conversely. (S is a regular open set if $S = \text{int } \overline{S}$ where int $\overline{S}$ is the largest open set in $\overline{S}$.) If S and T are regular open sets such that $A(S) = A(T)$, then $\overline{S} = \overline{T}$ by (3) and hence $S = \text{int } (\overline{S}) = \text{int } (\overline{T}) = T$. We have:

LEMMA 1. *The lattice of annulets of K is anti-isomorphic to the lattice of regular open sets in M.*

LEMMA 2. *The following are equivalent*:

(1) *The sum of two annulets is an annulet.*

(2) *Every annulet is generated by an idempotent.*

(3) *Every regular open set in M is closed.*

PROOF. Assume (1) and let $A(S)$ be any annulet, where S is a regular open set. Let $T = \overline{S}'$. Then T is a regular open set such that $S \cap T = 0$. Now $A(S) \cup A(T)$ (the smallest annulet containing $A(S)$ and $A(T)) = A(S) + A(T)$ since this is an annulet by (1). But $A(S) \cup A(T) = A(S \cap T) = A(0) = K$ by Lemma 1. Since $A(S) \cap A(T) = 0$ we have $A(S) \oplus A(T) = K$, and it follows as in Theorem 3 of [**5**] that $A(S) = eK$ where e is the characteristic function of S'.

Now assume (2), let S be a regular open set, and $A(S)$ the corresponding annulet. Then $A(S) = eK$, so $N(e) = N(eK) = N[A(S)] = \overline{S}$. Hence $\overline{S}$ is open and closed. Then $S = \text{int } \overline{S} = \overline{S}$, and S is closed.

Now assume (3) and let S, T be regular open sets ($S \cap T$ is also regular). Since S is open and closed, $A(S)$ is generated by the characteristic function of S'. Then $A(T)$, $A(S \cap T)$ are similarly generated by idempotents, and the proof can now be completed as in the proof of Theorem 2 of [**5**] to show $A(S) + A(T)$ is the annulet $A(S \cap T)$.

LEMMA 3. *The following statements are equivalent*:

(1) *Every nonzero closed ideal of K contains a minimal ideal.*

(2) *For each $k \neq 0$, there exists a minimal projection e such that $ke \neq 0$.*

(3) *The space M contains a dense subset of isolated points.*

PROOF. Assume (1) and let $k \in K$, $k \neq 0$. Let $J = \overline{Kk} > 0$. Then J contains a minimal ideal of form Ke for a projection e [**4**, p. 64] since any idempotent in $K = C(M)$ is obviously self-adjoint. Now Ke is a simple commutative ring with identity, hence a field. By the Gelfand-Mazur theorem Ke is isomorphic to the complex numbers, so that e is a minimal projection. If $ke = 0$, $(Kk)e = 0$, $Je = 0$, $e^2 = e = 0$, a contradiction, so that (2) follows.

The fact that (2) implies (3) is proven in Theorem 2 of [**3**].

Now assume (3), and let J be a closed ideal $\neq 0$ so that $k \neq 0$, $k \in J$. If $ke \neq 0$ for some minimal projection e, then $ke = \lambda e$ for complex λ, $e=\lambda^{-1}ke\in J$ and J contains the ideal Ke which is obviously minimal, since it is a field. Assume if possible that $ke=0$ for all minimal projections e. Let X_0 be the dense set of isolated points. Each minimal projection e is the characteristic function of a point of X_0 [**3**, Theorem 2]. Hence $k(x_0)=0$ for all $x_0\in X_0$, so that $k\equiv 0$, since X_0 is dense in M. This contradicts $k\neq 0$ in K, and completes the proof.

Lemmas 2 and 3 and the characterization of [**5**] give us:

THEOREM 4. *K is equivalent to $B(X)$ if and only if*

(1) *The structure space M contains a dense set X_0 of isolated points.*

(2) *Every regular open set in M is closed.*

3. In [**3**] it was shown that $K=C(M)$ is equivalent to $B(X)$ if and only if either of the following sets of conditions is satisfied:

A. (1) M contains a dense subset X_0 of isolated points.

(2) If $Q\leqq X_0$, then there exists an open and closed subset S of M such that $Q\leqq S$ and $S\cap X_0=Q$.

B. (1) For each $k\neq 0$ in K there exists a minimal projection e with $ke\neq 0$.

(2) For each subcollection A of the collection P of minimal projections, there exists a projection e_A such that $e_Ae_p=e_p$ for $p\in A$ and $e_Ae_p=0$ for $p\notin A$.

The conditions of A are equivalent to those of B directly. This is essentially proved in Theorem 2 of [**3**]. That the (1)'s are equivalent is pointed out in our Lemma 3. The (2)'s, for example, are each obviously equivalent to the statement: If A is a subset of X_0, there exists a function $e(x)$ in $C(M)$ such that $e(x)=1$ if $x\in A$, and $e(x)=0$ if $x\in X_0-A$.

Now assume that the conditions of Theorem 4 hold and let $Q\leqq X_0$. Then Q is open and $Q\leqq \operatorname{int}\bar{Q}\leqq\bar{Q}$. But since each subset of X_0 is open (as a union of points which are open) it is also closed in the relative topology of X_0. Hence $Q=X_0\cap\bar{Q}$ and this implies $Q=X_0\cap\operatorname{int}\bar{Q}$. Then $S=\operatorname{int}\bar{Q}$ is open and closed (being a regular open set) and the conditions of A are satisfied.

Now assume the conditions of A, and let S be a regular open set in M. Let Q be the complement of $X_0\cap S$ in X_0. Then $(\overline{X_0\cap S})\cup\bar{Q}$ $=\bar{X}_0=M$. Now by (2) of A, there exists an open and closed set T such that $X_0\cap S\leqq T$, but $Q\leqq T'$. Thus $\overline{X_0\cap S}\leqq T$, $\bar{Q}\leqq T'$, and $(\overline{X_0\cap S})\cap\bar{Q}=0$, and $\overline{X_0\cap S}$ and $\bar{Q}$ are open, as well as closed. Now since $S\cap Q=0$ and S is open, $S\cap\bar{Q}=0$. Since $\bar{Q}$ is open $\bar{S}\cap\bar{Q}=0$. Hence $\bar{S}\leqq\bar{Q}'=\overline{X_0\cap S}$. Thus $\bar{S}=\overline{X_0\cap S}$ and $\bar{S}$ is open. Then $S=\operatorname{int}(\bar{S})$ $=\bar{S}$, and S is closed. Hence the conditions of Theorem 4 are equivalent to A and B.

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LIE SIMPLICITY OF A SPECIAL CLASS OF ASSOCIATIVE RINGS[1]

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Given an associative ring A, by introducing a new multiplication we can form from it a new ring called the Lie ring of A. This multiplication is defined by $[a, b]=ab-ba$ for all $a, b\in A$. If U is an additive subgroup of A and if for arbitrary $u\in U$, $a\in A$, $ua-au\in U$, then U is said to be a Lie ideal of A. If X, Y are additive subgroups of A then by $[X, Y]$ we mean the additive subgroup generated by all the elements $xy-yx$, where $x\in X$, $y\in Y$. An additive subgroup U of $[A, A]$ is said to be a proper Lie ideal of $[A, A]$ if $U\neq[A, A]$ and if $[U, [A,A]]\subset U$.

In [4], Herstein proved that if A is a simple ring of characteristic not 2 or 3, and if U is a proper Lie ideal of $[A, A]$, then U is contained in Z, the center of A. In this paper we settle the question in the open case where A is a simple ring of characteristic 2 or 3. The above theorem becomes sharpened to:

Theorem 1. *If A is a simple ring and if U is a proper Lie ideal of $[A, A]$, then U is contained in the center of A, except for the case where A is of characteristic 2 and 4 dimensional over its center, a field of characteristic 2.*

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[1] The results of this paper will comprise the beginning portion of a thesis, which will be presented to the Faculty of the Graduate School of the University of Pennsylvania in partial fulfillment of the requirements for the degree of Doctor of Philosophy.