A NOTE ON THE ALGEBRA OF BOUNDED FUNCTIONS. II

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1. Let $K$ be a commutative $B^*$-algebra with identity $1$ (with $\|k^*k\| = \|k\|^2$ for all $k \in K$, and $\|1\| = 1$). Then $K$ is equivalent (isomorphic in a norm and * preserving manner) to the algebra $C(M)$ of all continuous complex-valued functions on the compact Hausdorff space $M$ (its structure space) [1; 2]. In [5] we have given necessary and sufficient conditions that $K$ be equivalent to $B(X)$, the ring of all bounded complex-valued functions on the discrete space $X$. These conditions were ideal-theoretic, involving the annihilets (anihilating ideals) of $K$, and did not depend on the representation $C(M)$. Using the representation $C(M)$ two further characterizations of $B(X)$ are given in [3], one involving the properties of the space $M$, and the other the notion of projection. The characterizations in [3] are derived independently of the one in [5], and in fact no attempt is made in [3] to relate directly the ideal-theoretic conditions with the notions used there. In this note, we show how the characterizations in [3] can be derived from the characterization in [5] by relating directly the ideal-theoretic properties of $K$ with the properties of the structure space $M$, and with the idea of projection. In particular, the lattice of annihilets of $K$ is anti-isomorphic to the lattice of regular open sets in $M$ (Lemma 1). Another characterization of $B(X)$ (Theorem 4) is a byproduct of our procedure.

2. The notation will follow that in [5]. If $G \subseteq K$ then $R(G)$ is the set of all functions $k \in K = C(M)$ such that $kg = 0$ for all $g \in G$. Such ideals were called annihilets. $N(G)$ is the set of $y \in M$ such that $g(y) = 0$ for all $g \in G$. If $S \subseteq M$, then $A(S)$ is the set of functions $f$ such that $f(x) = 0$ for all $x \in S$. Since $N(G) = \bigcap_{g \in G} N(g)$, it is a closed set. Now by following the arguments of Lemma 1 of [5], and using the fact that if $O$ is open in a compact space and $x \in O$, there exists a function $f \in K$ with $f(x) = 1$ and $f(O') = 0$, we have

- (1) $R(G) = A[N(G)]'$ for $G \subseteq K$;
- (2) $R[A(S)] = A(S')$ if $S$ is a closed subset of $M$;
- (3) $N[A(S)] = S$ if $S$ is a closed subset of $M$.

Now (1) and (2) show that an annihilet of $K$ is the set of all functions vanishing on an open set of $M$ and conversely. Since $A(S) = A(S)$

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and $S \leq \text{int } S \leq \bar{S}$ for any open set $S$, an annulet is the set of functions vanishing on a regular open set and conversely. ($S$ is a regular open set if $S = \text{int } \bar{S}$ where $\text{int } \bar{S}$ is the largest open set in $\bar{S}$.) If $S$ and $T$ are regular open sets such that $A(S) = A(T)$, then $\bar{S} = \bar{T}$ by (3) and hence $S = \text{int } (\bar{S}) = \text{int } (\bar{T}) = T$. We have:

**Lemma 1.** The lattice of annulets of $K$ is anti-isomorphic to the lattice of regular open sets in $M$.

**Lemma 2.** The following are equivalent:

1. The sum of two annulets is an annulet.
2. Every annulet is generated by an idempotent.
3. Every regular open set in $M$ is closed.

**Proof.** Assume (1) and let $A(S)$ be any annulet, where $S$ is a regular open set. Let $T = S'$. Then $T$ is a regular open set such that $S \cap T = 0$. Now $A(S) \cup A(T)$ (the smallest annulet containing $A(S)$ and $A(T)$) $= A(S) + A(T)$ since this is an annulet by (1). But $A(S) \cup A(T) = A(S \cap T) = A(0) = K$ by Lemma 1. Since $A(S) \cap A(T) = 0$ we have $A(S) \oplus A(T) = K$, and it follows as in Theorem 3 of [5] that $A(S) = eK$ where $e$ is the characteristic function of $S'$.

Now assume (2), let $S$ be a regular open set, and $A(S)$ the corresponding annulet. Then $A(S) = eK$, so $N(e) = N(eK) = N[A(S)] = \bar{S}$. Hence $\bar{S}$ is open and closed. Then $S = \text{int } \bar{S} = \bar{S}$, and $S$ is closed.

Now assume (3) and let $S, T$ be regular open sets ($S \cap T$ is also regular). Since $S$ is open and closed, $A(S)$ is generated by the characteristic function of $S'$. Then $A(T), A(S \cap T)$ are similarly generated by idempotents, and the proof can now be completed as in the proof of Theorem 2 of [5] to show $A(S) + A(T)$ is the annulet $A(S \cap T)$.

**Lemma 3.** The following statements are equivalent:

1. Every nonzero closed ideal of $K$ contains a minimal ideal.
2. For each $k \neq 0$, there exists a minimal projection $e$ such that $ke \neq 0$.
3. The space $M$ contains a dense subset of isolated points.

**Proof.** Assume (1) and let $k \in K$, $k \neq 0$. Let $J = Ke$ be a projection $e$ [4, p. 64] since any idempotent in $K = C(M)$ is obviously self-adjoint. Now $Ke$ is a simple commutative ring with identity, hence a field. By the Gel-fand-Mazur theorem $Ke$ is isomorphic to the complex numbers, so that $e$ is a minimal projection. If $ke = 0$, $(Kk)e = 0$, $Je = 0$, $e^2 = e = 0$, a contradiction, so that (2) follows.

The fact that (2) implies (3) is proven in Theorem 2 of [3].

Now assume (3), and let $J$ be a closed ideal $\neq 0$ so that $k \neq 0$, $k \in J$. If $ke \neq 0$ for some minimal projection $e$, then $ke = \lambda e$ for com-
plex \( \lambda, e = \lambda^{-1}ke \in J \) and \( J \) contains the ideal \( Ke \) which is obviously minimal, since it is a field. Assume if possible that \( ke = 0 \) for all minimal projections \( e \). Let \( X_0 \) be the dense set of isolated points. Each minimal projection \( e \) is the characteristic function of a point of \( X_0 \) [3, Theorem 2]. Hence \( k(x_0) = 0 \) for all \( x_0 \in X_0 \), so that \( k = 0 \), since \( X_0 \) is dense in \( M \). This contradicts \( k \neq 0 \) in \( K \), and completes the proof.

Lemmas 2 and 3 and the characterization of [5] give us:

**Theorem 4.** \( K \) is equivalent to \( B(X) \) if and only if

1. The structure space \( M \) contains a dense set \( X_0 \) of isolated points.
2. Every regular open set in \( M \) is closed.

3. In [3] it was shown that \( K = C(M) \) is equivalent to \( B(X) \) if and only if either of the following sets of conditions is satisfied:
   
   A. (1) \( M \) contains a dense subset \( X_0 \) of isolated points.
   (2) If \( Q \subseteq X_0 \), then there exists an open and closed subset \( S \) of \( M \) such that \( Q \subseteq S \) and \( S \cap X_0 = Q \).

   B. (1) For each \( k \neq 0 \) in \( K \) there exists a minimal projection \( e \) with \( ke \neq 0 \).
   (2) For each subcollection \( A \) of the collection \( P \) of minimal projections, there exists a projection \( e_A \) such that \( e_A e_p = e_p \) for \( p \in A \) and \( e_A e_p = 0 \) for \( p \notin A \).

   The conditions of A are equivalent to those of B directly. This is essentially proved in Theorem 2 of [3]. That the (1)'s are equivalent is pointed out in our Lemma 3. The (2)'s, for example, are each obviously equivalent to the statement: If \( A \) is a subset of \( X_0 \), there exists a function \( e(x) \) in \( C(M) \) such that \( e(x) = 1 \) if \( x \in A \), and \( e(x) = 0 \) if \( x \in X_0 - A \).

Now assume that the conditions of Theorem 4 hold and let \( Q \subseteq X_0 \). Then \( Q \) is open and \( Q \subseteq int Q \subseteq \overline{Q} \). But since each subset of \( X_0 \) is open (as a union of points which are open) it is also closed in the relative topology of \( X_0 \). Hence \( Q = X_0 \cap \overline{Q} \) and this implies \( Q = X_0 \cap int \overline{Q} \). Then \( S = int \overline{Q} \) is open and closed (being a regular open set) and the conditions of \( A \) are satisfied.

Now assume the conditions of \( A \), and let \( S \) be a regular open set in \( M \). Let \( Q \) be the complement of \( X_0 \cap S \) in \( X_0 \). Then \( \overline{X_0 \cap S} \cap \overline{Q} = \overline{X_0} = M \). Now by (2) of \( A \), there exists an open and closed set \( T \) such that \( X_0 \cap S \subseteq T \), but \( Q \subseteq T' \). Thus \( \overline{X_0 \cap S} \subseteq T \), \( \overline{Q} \subseteq T' \), and \( (X_0 \cap S) \cap \overline{Q} = 0 \), and \( \overline{X_0 \cap S} \) and \( \overline{Q} \) are open, as well as closed. Now since \( S \cap Q = 0 \) and \( S \) is open, \( S \cap \overline{Q} = 0 \). Since \( \overline{Q} \) is open \( S \cap \overline{Q} = 0 \). Hence \( S \subseteq \overline{Q'} = \overline{X_0 \cap S} \). Thus \( S = X_0 \cap S \) and \( S \) is open. Then \( S = int \overline{S} = \overline{S} \), and \( S \) is closed. Hence the conditions of Theorem 4 are equivalent to \( A \) and \( B \).
LIE SIMPLICITY OF A SPECIAL CLASS OF ASSOCIATIVE RINGS

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Given an associative ring $A$, by introducing a new multiplication we can form from it a new ring called the Lie ring of $A$. This multiplication is defined by $[a, b] = ab - ba$ for all $a, b \in A$. If $U$ is an additive subgroup of $A$ and if for arbitrary $u \in U$, $a \in A$, $ua - au \in U$, then $U$ is said to be a Lie ideal of $A$. If $X, Y$ are additive subgroups of $A$ then by $[X, Y]$ we mean the additive subgroup generated by all the elements $xy - yx$, where $x \in X, y \in Y$. An additive subgroup $U$ of $[A, A]$ is said to be a proper Lie ideal of $[A, A]$ if $U \neq [A, A]$ and if $[U, [A, A]] \subseteq U$.

In [4], Herstein proved that if $A$ is a simple ring of characteristic not 2 or 3, and if $U$ is a proper Lie ideal of $[A, A]$, then $U$ is contained in $Z$, the center of $A$. In this paper we settle the question in the open case where $A$ is a simple ring of characteristic 2 or 3. The above theorem becomes sharpened to:

**Theorem 1.** If $A$ is a simple ring and if $U$ is a proper Lie ideal of $[A, A]$, then $U$ is contained in the center of $A$, except for the case where $A$ is of characteristic 2 and 4 dimensional over its center, a field of characteristic 2.

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1 The results of this paper will comprise the beginning portion of a thesis, which will be presented to the Faculty of the Graduate School of the University of Pennsylvania in partial fulfillment of the requirements for the degree of Doctor of Philosophy.