ON COMMUTATORS AND JACOBI MATRICES\textsuperscript{1}

C. R. PUTNAM

1. All operators in this paper are bounded linear transformations on a Hilbert space. The commutator $C$ of two operators $A$ and $B$ is defined by

\begin{equation}
C = AB - BA.
\end{equation}

The closure, $W$, of the set of values $(Cx, x)$ when $\|x\| = 1$ is a closed convex set (Hausdorff, cf. [8, p. 34]). A complex number $z$ will be said to belong to the interior of $W$ if $z$ is in $W$ and if one of the following conditions holds: (i) If $W$ is two-dimensional, then $z$ does not lie on the boundary of $W$; (ii) If $W$ is a line segment, then $z$ is not an end point; (iii) $W$ consists of $z$ alone.

It was shown in [4] that if $A$ (or $B$) is normal, or even semi-normal, so that $AA^* - A^*A$ is semi-definite, then 0 belongs to $W$, but is not necessarily in the interior of $W$. (That, for arbitrary $A$ and $B$, in general 0 need not even belong to $W$ was shown in [2].) In fact, if $A = (a_{ij})$ is defined by $a_{i,i+1} = 1$, $a_{ij} = 0$ if $j \neq i + 1$, then $C = AA^* - A^*A = (c_{ij})$ is the self-adjoint matrix all elements of which are zero except $c_{11} = 1$. Consequently, $C \geq 0$ with a spectrum consisting of $\lambda = 0$, 1; hence $W$ is the segment $0 \leq \lambda \leq 1$ and 0 is not in the interior of $W$. Moreover, the above $C$ can also be expressed by $C = DA^* - A^*D$ where $D$ is the (self-adjoint) Jacobi matrix $A + A^*$; cf. § 5 below.

The problem to be considered in the present paper is that of determining a sufficient condition guaranteeing that 0 belongs to the interior of $W$. Such a condition yields information concerning the spec-

\textsuperscript{1} This work was supported by the National Science Foundation research grant NSF-G-481.

Received by the editors November 25, 1955.
trum of $C$, at least if $C$ is normal (since $W$ is then the least convex set containing the spectrum of $C$). Concerning the possible spectra for commutators (not necessarily normal) see [3, p. 198].

2. There will be proved the following principal lemma. Let $A$ be normal with the spectral resolution

\[ A = \int z dK(z). \]

Let $S$ be an arbitrary measurable (here, and in the sequel, “measurable” implies “measurable with respect to $K(z)$”) set in the complex plane with a covering $\{\Delta_1, \Delta_2, \Delta_3, \cdots\}$ by pair-wise disjoint measurable sets $\Delta_n$ of diameter $d_n$. Suppose that $H=C+C^*$ is non-negative definite with the self-adjoint square root $H^{1/2}$. Then, if $x$ is any element of Hilbert space, there holds the inequality

\[ \left\| H^{1/2} \int_S dKx \right\|^2 \leq 4\|B\|\|x\|^2 \left( \sum_n d_n \right). \]

Moreover, if $A$ is self-adjoint or unitary, then

\[ \left\| H^{1/2} \int_S dKx \right\|^2 \leq 4\|B\|\|x\|^2 \text{ (meas } S), \]

where the measure refers to the ordinary one-dimensional Lebesgue measure on a segment or a circle according as $A$ is self-adjoint or is unitary.

Proof of the lemma. Let $\Delta$ denote an arbitrary measurable set in the complex plane and let $\Delta K = \int \Delta dK$. Multiplication by $\Delta K$ on the left and on the right of the equation (1) yields

\[ \Delta K \Delta K = \int_\Delta zdK \Delta K - \Delta K \int_\Delta zdK \]

\[ = \int_\Delta (z - z_0)dK \Delta K - \Delta K \int_\Delta (z - z_0)dK, \]

where $z_0$ is an arbitrary constant. (Cf. [1] and [5] for calculations of a somewhat similar nature.) It is clear that

\[ (\Delta K x, C \Delta K x) = (\Delta K x, \int_\Delta (z - z_0)dK \Delta K x) \]

\[ - (\Delta K x, \Delta K \int_\Delta (z - z_0)dK x). \]
If adjoints are taken in (5), a relation (6') similar to (6) exists for $C^*$. Addition of (6) and (6') followed by a majorization leads to $(\Delta Kx, H\Delta Kx) \leq 4\|B\|\|\Delta Kx\|^2d$, where $d$ is the diameter of $\Delta$. Consequently, 

\[
\|H^{1/2}\Delta Kx\| \leq 2\|B\|^{1/2}\|\Delta Kx\|d^{1/2}.
\]

Next, by (7),

\[
\left\| H^{1/2} \int_S dKx \right\| \leq \sum \left\| H^{1/2} \int_{S\Delta_n} dKx \right\|
\leq \sum 2\|B\|^{1/2} \int_{S\Delta_n} dKx \| (\text{diam } S\Delta_n)^{1/2}.
\]

In view of the inequalities $\sum_n \| \int_{S\Delta_n} dKx \|^2 = \int_{Sd} \|Kx\|^2 \leq \|x\|^2$ and diam $S\Delta_n \leq d_n$, an application of the Schwarz inequality to (8) now yields the desired relation (3). Relation (4) is readily verified to be a consequence of (3) (if one chooses the $\Delta_n$ for instance to be rectangles) and the proof of the lemma is now complete.

3. As a consequence of the lemma, there will be derived in this section the following

**Theorem.** (i) Suppose that $A$ is normal with the spectral resolution (2) and that

\[
I = \int_S dK,
\]

for a measurable set $S$. If $S$ can be covered by a sequence of pair-wise disjoint measurable sets $\Delta_n$ for which $\sum_n (\text{diam } \Delta_n)$ can be made arbitrarily small, then 0 belongs to the interior of the set $W$ associated with $C$ of (1).

(ii) Moreover, if $A$ is self-adjoint or unitary, and if there exists a set $S=Z$ of (one-dimensional) measure zero for which (9) holds, then 0 belongs to the interior of the set $W$.

**Proof of the Theorem.** Suppose, if possible, that 0 is not in the interior of $W$. Then choose $\theta$ so that the set $W_{\theta} (= W \exp (i\theta))$ belonging to $C = A(B \exp (i\theta)) - (B \exp (i\theta))A (= C \exp (i\theta))$ lies entirely in the half-plane $R(z) \geq 0$. (Cf. [4, p. 129].) It is clear that $H_{\theta} = C_{\theta} + C_{\theta}^*$ $\geq 0$ and that $\|B \exp (i\theta)\| = \|B\|$. Consequently an application of (3) of the lemma shows that $\|H_{\theta}^{1/2}x\|^2 \leq 4\|B\| \|x\|^2(\sum_n d_n)$. Hence, in view of the hypothesis, $\|H_{\theta}^{1/2}x\| = 0$ for every $x$, that is, $H_{\theta}^{1/2}$ (hence $H_{\theta}$) is the zero operator. Therefore

\[
C \exp (i\theta) = -(C \exp (i\theta))^*.
\]
thus $C \exp (i\theta + i\pi/2)$ is self-adjoint. Since $W_{\theta+\pi/2}$ is therefore a segment of the real axis, the set $W$ is a segment containing the origin. If however, 0 is not in the interior of this segment, there exists an angle $\phi$ for which the set $W_\phi$ belonging to $C \exp (i\phi)$ lies on the real axis with left end point at 0. In view of relation (10), which now holds for $\theta = \phi$, and the fact that $C \exp (i\phi)$ is self-adjoint, it follows that $C \exp (i\phi) = 0$, hence $C = 0$, in contradiction to the assumption that 0 is not in the interior of $W$. This completes the proof of (i) of the theorem.

The proof of part (ii) of the theorem is similar if one notes that (4) now applies to the set $S = Z$ and $\text{meas} \ Z = 0$.

4. Various corollaries of the above theorem will be collected in this section.

**Corollary 1.** If $A$ (or $B$) is normal with a pure point spectrum then 0 is in the interior of the set $W$.

The proof is clear from part (i) of the theorem if it is noted that $S$ can be chosen to be a denumerable point set.

**Corollary 2.** If $A$ (or $B$) is self-adjoint or unitary with spectrum of (one-dimensional) measure zero then 0 is in the interior of $W$.

The proof follows from part (ii) of the theorem if it noted that (9) must hold when $S$ is the spectrum of $A$.

**Corollary 3.** If $C = AA^* - A^*A$ is semi-definite, then either (i) $C = 0$ or (ii) if $A + A^*$ has the spectral resolution $\int \lambda dE(\lambda)$, then $\int \lambda dE < 1$ for every (one-dimensional) zero set $Z$ on the real axis.

It is clear that

$$C = (A + A^*)A^* - A^*(A + A^*),$$

so that $A + A^*$ can be identified with $A$ of the theorem. If now (ii) of Corollary 3 fails to hold, so that $\int \lambda dE = I$ for some zero set $Z$, then part (ii) of the theorem implies that 0 belongs to the interior of $W$. Since however the end points of $W$ belong to the spectrum of the (semi-definite) operator $C$, then $W$ must reduce to the single point 0. Therefore $C = 0$; that is, (i) of Corollary 3 holds.

5. On the spectra of Jacobi matrices. That $C$ of Corollary 3 and (11) need not be zero was shown by the example in §1 where $A + A^* = D = (d_{ij})$ is the Jacobi matrix with $d_{i,i+1} = d_{i+1,i} = 1, d_{ii} = 0$ otherwise. (For a discussion of this type of Jacobi matrix (i.e., one in which the diagonal elements are 0) see [8, p. 236].
More generally if $A = (a_{ij})$ is defined by $a_{i,j+1} = b_i$ and $a_{ij} = 0$ for $j \neq i+1$, then $A + A^* = D = (d_{ij})$ is the self-adjoint matrix with $d_{i,i+1} = b_i$, $d_{i+1,i} = b^*_i$, and $d_{ij} = 0$ otherwise. Moreover it is easily verified that $C = DA^* - A^*D$ is the diagonal matrix with diagonal elements $|b_1|^2$, $|b_2|^2$, $|b_3|^2$, ..., $|b_i|^2$, ..., and if the $b_i$ satisfy

$$0 < |b_1| \leq |b_2| \leq |b_3| \leq \cdots,$$

then $D$ is a Jacobi matrix and $C (\neq 0)$ is non-negative definite. Thus, case (ii) of Corollary 3 applies and if $D = \int \lambda dE(\lambda)$ then $\int_Z dE \leq 1$ for every zero set $Z$ of the $\lambda$ axis. (In particular, when (12) holds, $D$ cannot have a pure point spectrum, nor can its spectrum be a zero set.)

REFERENCES

5. ———, On normal operators in Hilbert space, ibid. vol. 73 (1951) pp. 357-362.