A NOTE ON THE SEPARATION OF CONNECTED SETS BY FINITE SETS

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A connected set K is said to be separated by a subset H of K if $K-H$ is not connected. J. R. Kline has shown that if n is an integer greater than two and the plane continuum M is separated by every subset of M consisting of n points, then M is separated by some set consisting of $n-1$ points [1, Theorem 5]. A stronger conclusion has been obtained by G. T. Whyburn for the case where M is a locally compact connected metric space. In fact, it follows from Whyburn's results that if every set consisting of n points separates the nondegenerate locally compact connected metric space M, then M is a Menger regular curve and contains uncountably many mutually exclusive pairs of points each pair of which separates M[2, p. 313]. It is the purpose of this note to present a proof of a related theorem for a connected topological space.

THEOREM. *If S is a nondegenerate connected topological space*[1] *and D is an open set such that each infinite subset of D contains a finite set that separates S, then some pair of points in D separates S.*

The following two lemmas will be used in the proof of this theorem.

LEMMA 1. *If S is a connected topological space, M_1 and M_2 are mutually exclusive closed sets such that M_2 does not separate S, and H is a connected subset of $S-(M_1+M_2)$ such that some open set contains M_1 and lies in $H+M_1$, then M_1+M_2 does not separate S.*

PROOF. Suppose $S-(M_1+M_2)$ is the sum of two mutually separated sets X and Y, where X contains the connected set H. Since some open set lies in $H+M_1$ and contains M_1, it follows that no point of M_1 is a limit point of Y. This leads to the contradiction that $S-M_2$ is the sum of the two mutually separated sets $X+M_1$ and Y.

LEMMA 2. *If D is an open set in a connected topological space S, L is a finite subset of D consisting of more than two points such that $S-L$ is the sum of two mutually separated sets H and K, and no subset of D with fewer points than L separates S, then for each point p of $D\cdot H$ the set $H+L-p$ is connected.*

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[1] The definition of a topological space given in [3] is used here.

PROOF. Suppose there is a point p of $D\cdot H$ such that $H+L-p$ is the sum of two mutually separated sets X and Y. Since $S-P$ is connected, it follows that both X and Y intersect L. Let n denote the number of points in L. Since $n>2$, it follows that one of the sets $X\cdot L+p$ and $Y\cdot L+p$ consists of less than n points. This involves a contradiction since each of these two subsets of D separates S.

PROOF OF THEOREM. Suppose that no pair of points in D separates S. Let L_1 be a subset of D such that (1) $S-L_1$ is the sum of two mutually separated sets H_1 and K_1 and (2) no set in D with fewer points than L_1 separates S. Let p_1 be a point of $K_1\cdot D$. By Lemma 2, $K_1+L_1-p_1$ is connected.

Let L_2 be a subset of $D\cdot H_1$ such that (1) $S-L_2$ is the sum of two mutually separated sets H_2 and K_2, where K_2 contains the connected set K_1+L_1, and (2) no set in $D\cdot H_1$ with fewer points than L_2 separates S. Let p_2 be a point of $D\cdot[K_2-(K_1+L_1)]$. By Lemma 2, $K_2+L_2-p_2$ is connected, and since $K_1+L_1-p_1$ is connected and K_1 is an open set lying in K_1+L_1, it follows from Lemma 1 that p_1+p_2 does not separate the connected set K_2+L_2.

Let L_3 be a subset of $D\cdot H_2$ such that (1) $S-L_3$ is the sum of two mutually separated sets H_3 and K_3, where K_3 contains the connected set K_2+L_2, and (2) no set in $D\cdot H_2$ with fewer points than L_3 separates S. Let p_3 be a point of $D\cdot[K_3-(K_2+L_2)]$. By Lemma 2, $K_3+L_3-p_3$ is connected, and since $K_2+L_2-(p_1+p_2)$ is connected and K_2 is an open set lying in K_2+L_2, it follows from Lemma 1 that $p_1+p_2+p_3$ does not separate the connected set K_3+L_3.

By continuing this process indefinitely, a sequence of distinct points $p_1, p_2, p_3, \cdots$ of D can be obtained such that, for each n, $p_1+p_2+\cdots+p_n$ does not separate the connected set K_n+L_n. Since each H_n+L_n is connected, it readily follows that for each n, $p_1+p_2+\cdots+p_n$ does not separate S. This leads to the contradiction that no finite subset of the infinite set $p_1+p_2+p_3+\cdots$ separates S.

COROLLARY. *If n is a positive integer and the nondegenerate connected topological space S is separated by every set consisting of n points, then each open set contains a pair of points that separates S.*

REFERENCES

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