A NOTE ON THE SEPARATION OF CONNECTED SETS BYFINITE SETS

C. E. BURGESS

A connected set $K$ is said to be separated by a subset $H$ of $K$ if $K - H$ is not connected. J. R. Kline has shown that if $n$ is an integer greater than two and the plane continuum $M$ is separated by every subset of $M$ consisting of $n$ points, then $M$ is separated by some set consisting of $n - 1$ points [1, Theorem 5]. A stronger conclusion has been obtained by G. T. Whyburn for the case where $M$ is a locally compact connected metric space. In fact, it follows from Whyburn’s results that if every set consisting of $n$ points separates the nondegenerate locally compact connected metric space $M$, then $M$ is a Menger regular curve and contains uncountably many mutually exclusive pairs of points each pair of which separates $M$ [2, p. 313]. It is the purpose of this note to present a proof of a related theorem for a connected topological space.

**Theorem.** If $S$ is a nondegenerate connected topological space and $D$ is an open set such that each infinite subset of $D$ contains a finite set that separates $S$, then some pair of points in $D$ separates $S$.

The following two lemmas will be used in the proof of this theorem.

**Lemma 1.** If $S$ is a connected topological space, $M_1$ and $M_2$ are mutually exclusive closed sets such that $M_2$ does not separate $S$, and $H$ is a connected subset of $S - (M_1 + M_2)$ such that some open set contains $M_1$ and lies in $H + M_1$, then $M_1 + M_2$ does not separate $S$.

**Proof.** Suppose $S - (M_1 + M_2)$ is the sum of two mutually separated sets $X$ and $Y$, where $X$ contains the connected set $H$. Since some open set lies in $H + M_1$ and contains $M_1$, it follows that no point of $M_1$ is a limit point of $Y$. This leads to the contradiction that $S - M_2$ is the sum of the two mutually separated sets $X + M_1$ and $Y$.

**Lemma 2.** If $D$ is an open set in a connected topological space $S$, $L$ is a finite subset of $D$ consisting of more than two points such that $S - L$ is the sum of two mutually separated sets $H$ and $K$, and no subset of $D$ with fewer points than $L$ separates $S$, then for each point $p$ of $D - H$ the set $H + L - p$ is connected.

Presented to the Society, September 2, 1955; received by the editors July 22, 1955 and, in revised form, January 16, 1956.

1 The definition of a topological space given in [3] is used here.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Proof. Suppose there is a point \( p \) of \( D \cdot H \) such that \( H + L - p \) is the sum of two mutually separated sets \( X \) and \( Y \). Since \( S - P \) is connected, it follows that both \( X \) and \( Y \) intersect \( L \). Let \( n \) denote the number of points in \( L \). Since \( n > 2 \), it follows that one of the sets \( X \cdot L + p \) and \( Y \cdot L + p \) consists of less than \( n \) points. This involves a contradiction since each of these two subsets of \( D \) separates \( S \).

Proof of theorem. Suppose that no pair of points in \( D \) separates \( S \). Let \( L_1 \) be a subset of \( D \) such that (1) \( S - L_1 \) is the sum of two mutually separated sets \( H_1 \) and \( K_1 \) and (2) no set in \( D \) with fewer points than \( L_1 \) separates \( S \). Let \( p_1 \) be a point of \( K_1 \cdot D \). By Lemma 2, \( K_1 + L_1 - p_1 \) is connected.

Let \( L_2 \) be a subset of \( D \cdot H_1 \) such that (1) \( S - L_2 \) is the sum of two mutually separated sets \( H_2 \) and \( K_2 \), where \( K_2 \) contains the connected set \( K_1 + L_1 \), and (2) no set in \( D \cdot H_1 \) with fewer points than \( L_2 \) separates \( S \). Let \( p_2 \) be a point of \( D \cdot [K_2 - (K_1 + L_1)] \). By Lemma 2, \( K_2 + L_2 - p_2 \) is connected, and since \( K_1 + L_1 - p_1 \) is connected and \( K_1 \) is an open set lying in \( K_1 + L_1 \), it follows from Lemma 1 that \( p_1 + p_2 \) does not separate the connected set \( K_2 + L_2 \).

Let \( L_3 \) be a subset of \( D \cdot H_2 \) such that (1) \( S - L_3 \) is the sum of two mutually separated sets \( H_3 \) and \( K_3 \), where \( K_3 \) contains the connected set \( K_1 + L_1 + L_2 \), and (2) no set in \( D \cdot H_2 \) with fewer points than \( L_3 \) separates \( S \). Let \( p_3 \) be a point of \( D \cdot [K_3 - (K_2 + L_2)] \). By Lemma 2, \( K_3 + L_3 - p_3 \) is connected, and since \( K_2 + L_2 - (p_1 + p_2) \) is connected and \( K_2 \) is an open set lying in \( K_2 + L_2 \), it follows from Lemma 1 that \( p_1 + p_2 + p_3 \) does not separate the connected set \( K_3 + L_3 \).

By continuing this process indefinitely, a sequence of distinct points \( p_1, p_2, p_3, \ldots \) of \( D \) can be obtained such that, for each \( n \), \( p_1 + p_2 + \cdots + p_n \) does not separate the connected set \( K_n + L_n \). Since each \( H_n + L_n \) is connected, it readily follows that for each \( n \), \( p_1 + p_2 + \cdots + p_n \) does not separate \( S \). This leads to the contradiction that no finite subset of the infinite set \( p_1 + p_2 + p_3 + \cdots \) separates \( S \).

Corollary. If \( n \) is a positive integer and the nondegenerate connected topological space \( S \) is separated by every set consisting of \( n \) points, then each open set contains a pair of points that separates \( S \).

References


University of Utah