P-RECURRENCE IN TOPOLOGICAL DYNAMICS

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It is the purpose of this paper to extend the notions of recurrence, both positive and negative [3, Chap. 7], to transformation groups whose phase groups are more general than the integers or the reals. The notion of regional recurrence is similarly treated and various applications to other topics of topological dynamics are given.

1. Definitions. The general reference for definitions is [3]. Throughout the paper \((X, T, \pi)\) will denote a transformation group for which \(X\) is a compact Hausdorff space. Whenever it is stated that \(X\) is a uniform space, it will be implicitly assumed that the (Hausdorff) topology of \(X\) is the one induced by the uniformity. It is further assumed that \(T\) is abelian. \(P\) and \(Q\) will be used to denote replete semigroups in \(T\), which are distinct from \(T\). Full use of all these hypotheses is not required in every theorem. For \(x \in X\) the \(P\)-limit set of \(x\), \(P_x\), is defined as in [1, 2.06]. Let \(E \subseteq T\), then \(E\) is \(P\)-extensive provided \(E \cap pP \neq \emptyset\) for each \(p \in P\). Let \(x \in X\), then \(x\) is \(\{P\text{-recurrent}\}\) \(\{P\text{-regionally recurrent}\}\) provided that for each neighborhood, \(U\), of \(x\) there exists a \(P\)-extensive set \(E\) in \(T\) such that \(r \in E\) implies \(\{xr \in U\} \subseteq U \cap U \neq \emptyset\). It is now clear that the restriction \(P = T\) is justified, for if we were to allow equality it would follow that each point \(x \in X\) is \(T\)-recurrent. This would make \(P\)-recurrence too weak a property to be of much interest. It is also clear that a point \(x \in X\) is recurrent if and only if \(x\) is \(P\)-recurrent for each replete semigroup \(P\) in \(T\). A similar statement holds for regional recurrence. \(T\) is said to be \(\{P\text{-regionally recurrent}\}\) \(\{\text{regionally recurrent}\}\) provided that for each \(x \in X\), \(x\) is \(\{P\text{-regionally recurrent}\}\) \(\{\text{regionally recurrent}\}\).

2. \(P\)-recurrence.

Lemma 1. Let \(x \in X\), let \(P\) be a replete semigroup in \(T\), then the following statements are pairwise equivalent:

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Throughout this paper braces are used to indicate alternative readings of the same definition or theorem. All words in the first sets of braces are to be read together, all in the second sets together, etc.

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(1) \( x \) is \( P \)-recurrent, i.e. if \( U \) is a neighborhood of \( x \), then there exists a \( P \)-extensive set \( E \subset T \) such that \( xE \subset U \).

(2) \( x \in P_x \).

(3) \( \text{Cl} (xP) = P_x \). [N.B. \( \text{Cl} (A) \) denotes the closure of the set \( A \).]

(4) \( \text{Cl} (xT) = \text{Cl} (xP) \).

(5) If \( U \) is a neighborhood of \( x \) and if \( p \in P \), then \( xpP \cap U \neq \emptyset \).

(6) If \( U \) is a neighborhood of \( x \) and if \( t \in T \), then \( xtP \cap U \neq \emptyset \).

(7) If \( U \) is a neighborhood of \( x \) then \( xT \subset UP^{-1} \).

(8) \( x \in \text{Cl} (xpP) \) for each \( p \in P \).

(9) \( x \in \text{Cl} (xtP) \) for each \( t \in T \).

(10) \( x \) is \( pP \)-recurrent for each \( p \in P \).

The proof is straightforward and is omitted. It is clear that if \( x \in X \) and if \( P \) and \( Q \) are replete semigroups in \( T \) such that \( P \subset Q \), then \( x \) is \{ \( P \)-recurrent \} \{ \( P \)-regionally recurrent \} implies \( x \) is \{ \( Q \)-recurrent \} \{ \( Q \)-regionally recurrent \}.

**Theorem 1.** Let \( x \in X \) and let \( x \) be \( P \)-recurrent, then \( xt \) is \( P \)-recurrent for each \( t \in T \).

**Proof.** \( x \in P_x \) by Lemma 1. Thus \( xt \in P_xt = P_xt \) by \([3, 6.34]\), and again by Lemma 1, \( xt \) is \( P \)-recurrent. This completes the proof.

We remark that if \( x \) is recurrent, then for any two replete semigroups \( P \) and \( Q \) in \( T \), \( P_x = Q_x \); however, the converse proposition is false, for consider the point \( y = \cdots 00100 \cdots \) in the bisequence space \([3, \text{Chap. 12}]\). Let \( X = \text{Cl} (yI) \), \( I \) the integers, then clearly \( X = yI \cup z \), where \( z = \cdots 00000 \cdots \), and \( P_x = z \) for each \( x \in X \) and each replete semigroup \( P \) in \( I \). We observe, however, that \( y \) is not recurrent.

In case \( T \) is generative \([1, 1.01]\) and separable there is an intimate connection between \( P \)-recurrence and \( P \)-asymptoticity \([1, 2.03]\), namely, the following statements are pairwise equivalent: (1) \( x \) is not \( P \)-recurrent, (2) \( x \notin P_x \), (3) \( x \uparrow_pP_x \), i.e. \( x \) is \( P \)-asymptotic to \( P_x \).

It is known \([3, 6.14]\) that if \( S \) is a closed syndetic subgroup of \( T \) and \( P \) a replete semigroup in \( T \), then \( P \cap S \) is a replete semigroup in \( S \). The following lemma asserts something more.

**Lemma 2.** Let \( S \) be a closed syndetic subgroup of \( T \), let \( P \) be a replete semigroup in \( T \), then \( P \neq T \) if and only if \( P \cap S \neq S \).

**Proof.** The sufficiency is obvious. We show the necessity. Assume the contrary, i.e. \( P \cap S = S \) and \( P \neq T \). Let \( K \subset T \) be compact such that \( T = SK \). Since \( K \) is compact, there exists \( t \in T \) such that \( tk \in P \). Thus \( P \supset PP \supset tKP \supset tKS \supset tT \supset T \), whence \( P = T \), contrary to hypothesis. This completes the proof.
Theorem 2. Let $S$ be a closed syndetic subgroup in $T$, let $P$ be a replete semigroup in $T$, then $E \subseteq S$ is $P \cap S$-extensive if and only if $E$ is $P$-extensive.

Proof. Let $E \subseteq S$ be $P$-extensive in $T$; then $E \cap pP \neq \emptyset$ for each $p \in P$. Then certainly $E \cap p(P \cap S) \neq \emptyset$ for each $p \in P \cap S$, whence $E$ is $P \cap S$-extensive.

Conversely, let $E \subseteq S$ be $P \cap S$-extensive in $S$, i.e. $E \cap p(P \cap S) \neq \emptyset$ for each $p \in P \cap S$. Let $q \in P$ and let $K \subseteq T$ be compact such that $T = SK$; then there exists $t \in T$ such that $tK \subseteq P^{-1}$. Thus $Skt = Skt = Tt = T$, and we may assume that $K \subseteq P^{-1}$. Since $q \in T$, $q = sp_0^{-1}$, where $s \in S$ and $p_0^{-1} \in K \subseteq P^{-1}$, whence $s = qp$. Now $\{s, e\}$ is a compact set in $S$, therefore there exists $r \in S$ such that $\{rs, r\} \subseteq P \cap S$. Then $E \cap rs(P \cap S) \neq \emptyset$ and a fortiori $E \cap rsP = E \cap rqP \neq \emptyset$, whence $E \cap qP \supset E \cap (rpP) \neq \emptyset$. Thus $E$ is $P$-extensive in $T$. This completes the proof.

Theorem 3 (Inheritance theorem). Let $S$ be a closed syndetic subgroup in $T$, let $T$ be locally compact, and let $P$ be a replete semigroup in $T$ then:

1. if $x \in X$, $x$ is $P \cap S$-recurrent if and only if $x$ is $P$-recurrent.

2. $S$ is pointwise $P \cap S$-recurrent if and only if $T$ is pointwise $P$-recurrent, i.e. each $x \in X$ is $P \cap S$-recurrent if and only if each $x$ is $P$-recurrent.

Proof. Let $E$ be $P$-extensive in $T$, $C$ compact in $T$ and $E \subseteq BC$. Let $t \in T$ such that $tC \subseteq P^{-1}$. Since we may assume that $e \in C$, we have $t \in P^{-1}$. Let $p \in P$, then $E \cap t^{-1}pP \neq \emptyset$; thus let $a \in E$ such that $a = t^{-1}pq$, where $q \in P$. We have also that $a = bc$, where $b \in B$ and $c \in C$. Thus $at = bct = btp_0^{-1}$, where $p_0^{-1} = tC \subseteq tC \subseteq P^{-1}$. Therefore $b = atp_0 = pqt^{-1}tp_0 = pqp_0 \in P$, whence $B \cap pP \neq \emptyset$. Thus $B$ is $P$-extensive. The conclusion follows from Lemma 2, Theorem 2 and [3, 3.36]. This completes the proof.

3. $P$-regional recurrence.

Lemma 3. Let $x \in X$, then the following statements are pairwise equivalent:

1. $x$ is $P$-regionally recurrent, i.e. if $U$ is a neighborhood of $x$, then there exists a $P$-extensive set $E$ in $T$ such that $r \in E$ implies $U \cap Ur \neq \emptyset$.

2. If $U$ is a neighborhood of $x$, then $U \cap UqP \neq \emptyset$ for each $q \in P$.

3. If $U$ is a neighborhood of $x$, then $x \in C \cap (UqP)$ for each $q \in P$.

4. If $U$ is a neighborhood of $x$ and if $t \in T$, then $U \cap UtqP \neq \emptyset$ for each $q \in P$. 

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5) If $U$ is a neighborhood of $x$ and if $t \in T$, then $x \in \text{Cl} \ (U_t q P)$ for each $q \in P$.

6) If $U$ is a neighborhood of $x$, then $xT \subseteq \text{Cl} \ (U_q P)$ for each $q \in P$.

7) If $U$ is a neighborhood of $x$, then $\text{Cl} \ (xT) \subseteq \text{Cl} \ (U_q P)$ for each $q \in P$.

The proof is straightforward and is omitted.

**Lemma 4.** Let $x \in X$ and let $P$ be a replete semigroup in $T$. Let $x$ be $P$-regionally recurrent, and let $y \in \text{Cl} \ (xT)$, then $y$ is $P$-regionally recurrent.

**Proof.** [3, 3.27].

**Theorem 4.** Let $x \in X$, then each point of $P_x$ is $P$-regionally recurrent.

**Proof.** Let $y \in P_x$, and let $U$ be a neighborhood of $y$, then $y \in \text{Cl} \ (xP) \subseteq \text{Cl} \ (xP)$ for each $p \in P$. Let $p_0 \in P$ such that $xp_0 \in U$. Now $y \in \text{Cl} \ (xp_0 P)$ for each $p \in P$; thus let $q_0 \in P$ such that $xp_0 q_0 \in U$. Let $E = P \subseteq P_{p_0} q_0$. We show that $E$ is $P$-extensive. Let $s \in P$, then there exists $q \in P$ such that $p_0^{-1} sp \in E$. Thus $p_0^{-1} q P \subseteq p_0^{-1} s(P_0 P) = sP$, and $E \cap sP \neq \emptyset$. We show that $y$ is $P$-regionally recurrent. Let $r \in E$, $r = p_0^{-1} q p_0$ for some $p \in P$; then since $xp_0 \in U$, $xp_0 r = xp_0 (p_0^{-1} q p_0) = xp_0 q_0 \in U$. Since $xp_0 q_0 \in U$, we have $U \cap Ur \neq \emptyset$. This completes the proof.

Although the following theorem is only a minor variation of [3, 7.15], it will be needed in the sequel, and had best be stated explicitly here.

**Theorem 5.** Let $T$ be generative, let $T$ be $P$-regionally recurrent and let $R$ be the set of all $P$-recurrent points of $X$. Then:

1) If $X$ is metrizable, $R$ is an invariant residual $G_\delta$ set in $X$.

2) If $X$ is complete, then $X = \text{Cl} \ (R)$.

**Proof.** Use [3, 3.31] and [3, 6.09].

**Theorem 6.** Let $A$ be the set of all $P$-regionally recurrent points of $X$, then

1) $A$ is nonvacuous, closed, and invariant in $X$.

2) If $T$ is generative and separable, then each point of $X - A$ is $P$-asymptotic to $A$.

**Proof.** (1) By [3, 6.34] $P_x \neq \emptyset$ for each $x \in X$, and by Theorem 4 $A \supseteq P_x$, whence $A \neq \emptyset$. Let $y \in \text{Cl} \ (A)$, and let $U$ be any open neighborhood of $y$, then $U \cap A \neq \emptyset$. Let $a \in U \cap A$, then since $U$ is open,
U is a neighborhood of a; and since a is P-regionally recurrent, there exists a P-extensive set $E \subseteq T$ such that $r \in E$ implies $Ur \cap U \neq \emptyset$. This, however, implies that y is P-regionally recurrent, whence $y \in A$, and $A$ is closed.

Finally let $y \in A$, then, by Lemma 4, yt is P-regionally recurrent for each $t \in T$, whence $A$ is invariant.

(2) Let $x \in X - A$, then since $P_2 \subseteq A$, the conclusion follows from [1, 2.12]. This completes the proof.

**Theorem 7.** Let $X$ be a complete metric space, let $T$ be generative, then $T$ is P-regionally recurrent if and only if the set of all P-recurrent points of $X$ is dense in $X$.

**Proof.** We remark that if $x$ is P-recurrent then $x$ is P-regionally recurrent. The result then follows from Theorem 1, Theorem 6(1), and Theorem 5.

**Theorem 8.** Let $S$ be a closed syndetic subgroup of $T$ and let $T$ be locally compact. Then $S$ is $P \cap S$-regionally recurrent if and only if $T$ is P-regionally recurrent.

**Proof.** Let $S$ be $P \cap S$-regionally recurrent. Let $x \in X$, and let $U$ be a neighborhood of $x$; then there exists a $P \cap S$-extensive set $E \subseteq S$ such that $r \in E$ implies $U \cap Ur \neq \emptyset$. However, by Theorem 2, $E$ is P-extensive, whence $T$ is P-regionally recurrent.

Conversely, let $T$ be P-regionally recurrent, let $x \in X$, and let $U$ be a neighborhood of $x$. Then there exists an open neighborhood $U_0$ of $x$ and a compact symmetric neighborhood $V$ of $e$ in $T$ such that $U_0 \cap U \neq \emptyset$.

Now there exists $t \in T$ such that $tV \subseteq P$, whence $t \in P$. Let $p \in P \cap S \subseteq P$, then $ptVP \subseteq P \subseteq P$ and $V(ptP) \cap S \subseteq P \cap S$. We proceed inductively: there exists $ptp_1 \in ptP$ such that $U_0 \cap U_0 \cdot ptp_1 \neq \emptyset$. Thus there exists $x_1 \in U_0$ such that $x_1 \cdot ptp_1 \in U_0$. There exists $U_1$, an open neighborhood of $x_1$, such that $U_1 \subseteq U_0$ and $U_1 \cdot ptp_1 \subseteq U_0$.

At the $i$th step: there exists $ptp_i+1 \in ptP$ such that $U_i \cap U_i \cdot ptp_i \neq \emptyset$. Thus there exists $x_{i+1} \in U_i$ such that $x_{i+1} \cdot ptp_{i+1} \subseteq U_i$. There exists $U_{i+1}$, an open neighborhood of $x_{i+1}$, such that $U_{i+1} \subseteq U_i$ and $U_{i+1} \cdot ptp_{i+1} \subseteq U_i$.

There exists $K \subseteq T$, compact, such that $T = SK$. For each $i = 1, 2, \ldots$, there exists $s_i \in S$ and $k_i \in K$ such that $ptp_i = s_i k_i$. By [3, 6.22] we can find $n$ positive integers, $i_1, i_2, \ldots, i_n (n \geq 1)$ such that $i_1 < i_2 < \ldots < i_n$ and $k_1 k_2 \ldots k_n \subseteq S V$. Since $U_i \cdot ptp_j \subseteq U_{i-1}$ for $j = 2, 3, \ldots, n$, and since $U_i \cdot ptp_i \subseteq U_0$, we have $U_i \cdot ptp_j \subseteq U_0$, and in general $U_{i+1} \cdot p_{i+1} p_i \subseteq U_0$. Also $U_{i+1} \subseteq U_0$,
whence \( U_0 \cap U_0 p n \in \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \in \cdot \i
invariant subsets of \( X \) on which \( T \) is \( P \)-regionally recurrent. The center of the transformation group \( (X, T, \tau) \) is defined as in \([3, 7.17]\).

**Theorem 10.** Let \( C_P \) be the \( P \)-center of \( (X, T, \tau) \), then the following are valid statements:

1. \( C_P \) is closed in \( X \).
2. \( C_P \) is contained in the set of all \( P \)-regionally recurrent points of \( X \).
3. If \( x \) is \( P \)-recurrent, \( C_P \supseteq P_x \).
4. \( C_P \neq \emptyset \).

**Proof.** (1) and (2) are clear.

(3) Let \( y \in P_x \). By Lemma 1, \( \text{Cl} (xT) = P_x \). Let \( U \) be a neighborhood of \( y \) in \( P_x \), then there exists \( t \in T \) such that \( xt \in U \). Let \( V \) be a neighborhood of \( xt \) in \( P_x \) such that \( V \subseteq U \) and let \( V = V' \cap P_x \), where \( V' \) is a neighborhood of \( xt \) in \( X \). By Theorem 1 \( xt \) is \( P \)-recurrent, thus there exists a \( P \)-extensive set \( E \subseteq T \) such that \( xtE \subseteq V' \). Let \( r \in E \), then \( xtr \in V' \), and since \( xt \in V' \), \( xtr \in V' \). Also since \( xt \), \( xtr \in P_x = \text{Cl} (xT) \), we have \( xt \in V' \cap P_x = V' \), and \( xtr \in V' \cap P_x = V_r \), whence \( \emptyset \neq V \cap V_r \subseteq U \cap U_r \). Thus \( P_x \) is a \( T \)-invariant subset of \( X \) on which \( P \) is \( P \)-regionally recurrent, and \( C_P \supseteq P_x \).

(4) We remark that \([3, 2.22]\) guarantees the existence of a recurrent point \( x \in X \). It is clear that \( x \) is recurrent if and only if \( x \) is \( P \)-recurrent for each replete semigroup \( P \) in \( T \). The result then follows from (3) above and \([3, 6.34]\). This completes the proof.

Another approach to the \( P \)-center is the following: if \( X \) is an invariant subset of \( X \), define \( Y^* \) to be the set of all points of \( Y \) at which the transformation group \( (Y, T, \tau) \) is \( P \)-regionally recurrent. For each ordinal \( \alpha \) define \( P^\alpha \) as follows:

1. \( P^\alpha = X \).
2. If \( \alpha \) is a succession ordinal, then \( P^\alpha = (P^\alpha - 1)^* \).
3. If \( \alpha \) is a limiting ordinal, then \( P^\alpha = \bigcap_{\beta < \alpha} P^\beta \).

Then:

1. There exists a least ordinal \( \gamma \) such that \( P^\gamma = P^\alpha \) for all \( \alpha > \gamma \).
2. If \( X \) is second countable, then \( \gamma \) is countable.
3. \( P^\gamma = C_P \).

We also remark that if \( C \) is the center of \( (X, T, \tau) \) and if \( \emptyset \) is the class of all replete semigroups in \( T \), then \( C \subseteq \bigcap_{P \in \emptyset} C_P \).

**Theorem 11.** Let \( X \) be a complete metric space and let \( T \) be generative, then the \( P \)-center of \( (X, T, \tau) \) coincides with the closure of the set of all \( P \)-recurrent points of \( X \).

**Proof.** Let \( R_P \) be the collection of all \( P \)-recurrent points of \( X \) and let \( R_{C_P} \) be the collection of all \( P \)-recurrent points of \( C_P \). By Theorem
6, $C_p = \text{Cl} (R_{Cp})$. We show $R_p = R_{Cp}$. Let $x \in R_{Cp}$ and let $V$ be a neighborhood of $x$ in $C_p$, then $V = V' \cap C_p$, where $V'$ is a neighborhood of $x$ in $X$. Since $x \in R_{Cp}$, there exists a $P$-extensive set $E$ in $T$ such that $xE \subset V \subset V'$, whence $x \in R_p$. Conversely, let $x \in R_p$, then since $x$ is $P$-recurrent, $T$ is pointwise $P$-regionally recurrent on $xT$, i.e. $(xT, T, \pi)$ is $P$-regionally recurrent. Hence $xT \subset C_p$, and $x \in R_{Cp}$. Thus $R_p = R_{Cp}$, and this completes the proof.

Let $S$ be a subset of $T$ and let $M \subset X$. The set $M$ is called $S$-incompressible provided $MS \subset M$ implies $MS = M$. We now explore the relation between $P$-recurrence and incompressibility.

**Theorem 12.** $T$ is pointwise $P$-recurrent if and only if each closed set in $X$ is $pP$-incompressible for each $p \in P$.

**Proof.** Let $T$ be pointwise $P$-recurrent, let $M \subset X$ be closed and let $p \in P$ such that $M_{p}P \subset M$. Let $x \in M$, then $x \in P \subset M_{p}P \subset M$. We remark that by Lemma 1, $x \in P_\epsilon = \text{Cl} (xP)$ and that by [3, 6.34] $P_\epsilon$ is invariant, consequently $x \in P_\epsilon = P_{xP} = (\text{Cl} (xP) \cup P_\epsilon = [\text{Cl} (xP) \cup P \subset \text{Cl} (M) \cup P_\epsilon = M_{p}P$, whence $x \in M_{p}P$ and $M \subset M_{p}P$. Thus $M = M_{p}P$.

Conversely, let each closed set of $X$ be $pP$-incompressible for each $p \in P$. Let $x \in X$, let $p \in P$ and let $M = \text{Cl} (x \cup xP)$. Then $M_{p}P \subset \text{Cl} (xP \cup P \subset M$ and by hypothesis $M_{p}P = \text{Cl} (xP) = M$. It follows that $x \in Cl (xP)$ for each $p \in P$ and by Lemma 1, $x$ is $P$-recurrent. This completes the proof.

**Theorem 13.** $T$ is pointwise $P$-recurrent if and only if each open set in $X$ is $p^{-1}P^{-1}$-incompressible for each $p \in P$.

**Proof.** Let $T$ be pointwise $P$-recurrent. We remark that by Lemma 1, $xT \subset U_{p^{-1}}$ for each neighborhood $U$ of $x$, and thus $xT = xT_{p^{-1}} \subset U_{p^{-1}}$ for each $p \in P$.

Let $M$ be an open set in $X$ and let $p \in P$ such that $M_{p^{-1}}P^{-1} \subset M$. Let $x \in M$, then there exists a neighborhood $U$ of $x$ such that $U \subset M$. Thus $x \in xT \subset U_{p^{-1}}P^{-1} \subset M_{p^{-1}}P^{-1}$, and $M \subset M_{p^{-1}}P^{-1}$, whence $M = M_{p^{-1}}P^{-1}$.

Conversely, let each open set of $X$ be $p^{-1}P^{-1}$-incompressible. Let $t \in T$, then since $T = P^{-1}P$, $t = r^{-1}q$, where $r, q \in P$; also let $p \in P$ and define $M = Ut \cup UP^{-1}$. Then $M_{q^{-1}P^{-1}} = Ut_{q^{-1}P^{-1}} \cup U_{q^{-1}P^{-1}} \subset M_{q^{-1}P^{-1}} \cup U_{q^{-1}P^{-1}} \subset U_{P^{-1}} \subset M$, and since by hypothesis $M_{q^{-1}P^{-1}} = M$ it follows that $M_{q^{-1}P^{-1}} = U_{P^{-1}} = M$, whence $Ut \subset U_{P^{-1}}$. Thus $xt \in Ut \subset U_{P^{-1}}$ for each $t \in T$ and $xT \subset U_{P^{-1}}$. By Lemma 1, $x$ is $P$-recurrent. This completes the proof.
Corollary. Let $P$ be a replete semigroup in $T$ and let each \{closed\} \{open\} set in $X$ be \{$pP$-incompressible\} \{$p^{-1}P^{-1}$-incompressible\} for each $p \in P$, then $C_P = X$.

Theorem 14. Let $X$ be a compact metric space, let $T$ be generative and let $P$ be a replete semigroup in $T$. Let $C$ be the center of $X$ and $C_P$ be the $P$-center. For $B \subset X$ and $x \in X$ define $P(x, B)$ as in [2, Definition 2.1], and let $N(B)$ be defined to be any open set containing $B$. Then if $x \in X$, $P(x, N(C_P)) = P(x, N(C)) = 1$.

Proof. Let $R$ be the set of all recurrent points of $X$, let $R_P$ be the set of all $P$-recurrent points of $X$, and let $A$ be the minimal center of attraction of $X$ [2, Definition 3.4]. If $x$ is recurrent, then clearly $x$ is $P$-recurrent, hence $R_P \supseteq R$ and by [2, Theorem 4.2], $R \supset A$. Furthermore by Theorem 11, $C_P = \text{Cl} (R_P)$ and $C = \text{Cl} (R)$, whence $C_P \supset C \supset A$ and it follows by [2, Theorem 3.6] that $P(x, N(C_P)) = P(x, N(C)) = 1$. This completes the proof.

Bibliography


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