In this note we prove the

**Theorem.** If in a division ring $D$ an element $a \in D$ has only a finite number of conjugates in $D$ then it has only one conjugate, that is, $a$ is in $Z$, the center of $D$.

This theorem, of course, generalizes the famous theorem of Wedderburn which asserts that a finite division ring is a commutative field; however, since Wedderburn's theorem is used in the proof it does not yield a new proof of the result of Wedderburn. We also exhibit two corollaries to the theorem which may be of some independent interest; the second of these extends the result that a polynomial over a field having more roots than its degree in some extension field must be identically zero to a suitable analogue when the roots lie in a division ring.

**Proof of the Theorem.** We use the following convention throughout: if $K$ is a division ring then $K'$ will be the group of its nonzero elements under the multiplication of $K$.

Let $a \in D$ have a finite number of conjugates in $D$. Thus if $N = \{x \in D \mid xa = ax\}$ then $N$ is a subdivision ring of $D$; moreover $N'$ is of finite index in $D'$. Thus $N'$ has a finite number of conjugates in $D'$. Consequently $N$ has a finite number of conjugates in $D$, say $N = N_1, N_2, \ldots, N_k$; of course these $N_i$'s are subdivision rings of $D$. Since the $N_i$'s are all of finite index in $D'$ and there are a finite number of them, their intersection, $T'$, is also of finite index in $D'$; in addition $T'$ is normal in $D'$. Thus $T$, the intersection of the $N_i$'s is a subdivision

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ring of $D$ invariant under all the inner automorphisms of $D$. By the Brauer-Cartan-Hua theorem [1] either $T = D$ or $T \subseteq Z$, the center of $D$. If $T = D$, then $N = D$ and so $a$ is in $Z$. So we consider the second possibility, namely $T \subseteq Z$. But since $T'$ is of finite index in $D'$, the fact that $T \subseteq Z$ implies that $Z'$ is of finite index in $D'$.

If $Z$ is a finite field then since $Z'$ is of finite index in $D'$ it follows that $D$ is a finite division ring, and so is commutative by Wedderburn's theorem.

So we suppose that $Z$ has an infinite number of elements. Consider the elements $a_0 = a$, $a_1 = a + z_1$, $\cdots$, $a_n = a + z_n$, $\cdots$ where the $z_i$ are an infinite number of distinct elements of $Z$. Since the index of $Z'$ in $D'$ is finite, for some $z_i \neq z_j$, $a_i$ and $a_j$ must be in the same coset of $Z'$; that is $a + z_i = z(a + z_j)$ where $z \in Z$. Since $z_i \neq z_j$, $z$ cannot be equal to 1; but then $(1 - z)a = zz_j - z_i$ and so is in $Z$. Since $1 - z$ is in $Z$ and is not 0 it has an inverse in $Z$, from which we deduce that $a \in Z$, proving the theorem.

**Corollary 1.** Let $D$ be a division ring with center $Z$ and suppose that $p(x) = \alpha_0x^n + \alpha_1x^{n-1} + \cdots + \alpha_n$ where the $\alpha_i$ are in $Z$, has one root in $D$ which is outside of $Z$. Then $p(x)$ has an infinite number of roots in $D$.

**Proof.** Let $a \in D$, $a \notin Z$ be a root of $p(x)$; then all the conjugates of $a$ in $D$ are also roots; since $a \notin Z$ it has an infinite number of conjugates, proving the theorem.

**Corollary 2.** Let $D$ be a division ring with center $Z$ and suppose that $p(x)$ is a polynomial of degree $n$ with coefficients in $Z$. If $p(x)$ has $n + 1$ roots in $D$ then it has an infinite number of roots in $D$.

**Proof.** $p(x)$ has at most $n$ roots in $Z$ since $Z$ is a field, thus since it has $n + 1$ roots in $D$, one of these roots must fall outside $Z$, so the corollary reduces to Corollary 1.

**References**


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