PARTIALLY BOUNDED CONTINUED FRACTIONS

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For each complex number sequence \( a \), \( f(a) \) denotes the continued fraction

\[
\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}
\]

The statement that \( f(a) \) is partially bounded\(^1\) means that the sequence \( a \) has a bounded infinite subsequence. If \( f(a) \) is partially bounded, the series \( \sum |b_p| \) diverges, where \( b_1 = 1, a_p = 1/b_pb_{p+1}, p = 1, 2, \ldots \), —a necessary condition for convergence of \( f(a) \).

Any continued fraction \( f(a) \) such that \( \sum |b_p| \) diverges is convergent provided its even and odd parts are absolutely convergent,\(^2\) i.e. provided the series \( \sum |f_{2p+2} - f_{2p}| \) and \( \sum |f_{2p+1} - f_{2p-1}| \) are convergent, where \( \{f_p\}_{p=1}^\infty \) is the sequence of approximants. The simple convergence of the even and odd parts of \( f(a) \), together with the divergence of \( \sum |b_p| \), is not sufficient for the convergence of \( f(a) \), (Theorem 3). However, the simple convergence of the even and odd parts of the partially bounded continued fraction \( f(a) \) is sufficient for the convergence of \( f(a) \). In fact, we have this theorem:

**Theorem 1.** Suppose there is a positive integer \( k \) such that the sub-

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\(^1\) \( f(a) \) is called bounded if the sequence \( a \) is bounded—a condition equivalent to the boundedness of a certain infinite matrix. Cf. H. S. Wall, Analytic theory of continued fractions, 1948, p. 110. (Referred to later on as AT.)

sequence \( \{f_p\}_{p=1}^{\infty} \) of the sequence of approximants of the partially bounded continued fraction \( f(a) \) is bounded. If the even (odd) part of \( f(a) \) converges and has the value \( v \), then there is an infinite subsequence of the sequence of approximants of the odd (even) part of \( f(a) \) which converges to \( v \).

Proof. Let \( A_p \) and \( B_p \) be the \( p \)th numerator and denominator of \( f(a) \), so that \( A_0 = 0, A_1 = 1, B_0 = 1, B_1 = 1 \),

\[
A_{p+1} = A_p + a_p A_{p-1}, \quad B_{p+1} = B_p + a_p B_{p-1}, \quad p = 1, 2, \ldots,
\]

and

\[
A_p B_{p+1} - A_{p+1} B_p = A_p B_{p+2} - A_{p+2} B_p = (-1)^{p+1} a_1 a_2 \cdots a_p, \quad p = 1, 2, \ldots.
\]

Therefore, if \( p \geq k \),

\[
f_p - f_{p+1} = \frac{(-1)^{p+1} a_1 a_2 \cdots a_p}{B_p B_{p+1}}, \quad f_p - f_{p+2} = \frac{(-1)^{p+1} a_1 a_2 \cdots a_p}{B_p B_{p+2}},
\]

and consequently\(^*\)

\[
-a_{p+2}(f_p - f_{p+2})(f_{p+1} - f_{p+3}) = (f_p - f_{p+1})(f_{p+2} - f_{p+3}),
\]

or

\[
\left| a_{p+2}(f_p - f_{p+2})(f_{p+1} - f_{p+3}) \right| = \left| (f_p - f_{p+1})(f_{p+2} - f_{p+3}) \right|.
\]

Suppose the even (odd) part of \( f(a) \) converges to \( v \) and that no infinite subsequence of the sequence of approximants of the odd (even) part of \( f(a) \) converges to \( v \). Then, there exists a positive number \( c \) and a positive integer \( m \) greater than \( k \) such that if \( p \) is a positive integer greater than \( m \), \( \left| f_p - f_{p+2} \right| \leq M \). Let \( M \) be a positive integer such that, if \( p \geq k \), \( \left| f_p - f_{p+2} \right| \leq M \), and let \( L \) be a positive integer such that, for each positive integer \( r \) there exists a positive integer \( s \) greater than \( r \) for which \( \left| a_{s+2} \right| \leq L \). There exists a positive integer \( N \) greater than \( m \) such that if \( p \) is an integer greater than \( N \), \( \left| f_p - f_{p+2} \right| < c/ML \) or \( \left| f_{p+1} - f_{p+3} \right| < c/ML \). Hence, there exists a positive integer \( s \) greater than \( N \) such that \( e \leq \left| a_{s+2}(f_s - f_{s+2})(f_{s+1} - f_{s+3}) \right| \leq LM \cdot (c/ML) = e \). This contradicts our supposition false and establishes the theorem.

\(* \) This formula expresses the fact that \( a_{p+3} \) is a cross-ratio of four successive approximants of \( f(a) \). Cf. Wall, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 578–592.
Theorem 2. If the even and odd parts of the partially bounded continued fraction \( f(a) \) converge, then \( f(a) \) converges.

This is an immediate corollary to Theorem 1.

Theorem 3. There exists a divergent continued fraction \( f(a) \) whose even and odd parts converge for which the series \( \sum |b_p| \) diverges.

Proof. Let \( a \) denote the sequence defined by \( a_p = (-1)^p(p+1)^2, \) \( p = 1, 2, \ldots \). Since \( |b_p b_{p+1}|^{1/2} = 1/(p+1) \leq (|b_p| + |b_{p+1}|)/2 \), the series \( \sum |b_p| \) diverges. The even part of \( f(a) \) is \(-f(c)/3\), where \( c \) is the sequence defined by \( c_p = p^2(2p+1)/(2p-1), \) \( p = 1, 2, \ldots \), so that the even part of \( f(a) \) converges and its value is a negative number. The odd part of \( f(a) \) is \( 1+2f(d)/3 \), where \( d \) is the sequence defined by \( d_p = (p+1)^2(2p+1)/(2p+3), \) \( p = 1, 2, \ldots \), so that the odd part of \( f(a) \) converges and its value is a number greater than 1. Hence \( f(a) \) diverges.

A corollary to Theorem 2 is the following theorem of Farinha. Let \( E \) denote a bounded closed set in the complex plane which does not contain 0. If the even and odd parts of \( f(a) \) converge uniformly for \( a \) in \( E \), then \( f(a) \) converges uniformly for \( a \) in \( E \).

As an application of Theorem 2, we shall prove the following extension of a theorem of Thron.

Theorem 4. Suppose \( r \) is a positive number not greater than 1 and \( s \) a positive number less than \( (1+r)^{-2} \). The continued fraction \( f(a) \) such that \( |a_{2p}| \leq r^2 \) and \( |1/a_{2p-1}| \leq (1+r)^{-2} - s, \) \( p = 1, 2, \ldots \), is convergent.

Proof. Since \( f(a) \) is partially bounded, it suffices to show that its even and odd parts converge.

The even part of \( f(a) \) may be written as

\[
\frac{i/a_1}{b_1 + is} + \frac{2}{c_1} \left( \frac{i}{b_2 + is} \right) + \frac{2}{c_2} \left( \frac{i}{b_3 + is} \right) - \cdots ,
\]

where

\[
b_p = i + \frac{i}{a_{2p-1}} + \frac{a_{2p-2}}{a_{2p-1}} - is, \quad c_p = -\frac{a_{2p}}{a_{2p+1}}, \quad p = 1, 2, \ldots, (a_0 = 0).
\]

\(^4\) AT, pp. 58-59.
For each positive integer $p$, $\beta_p = \Im b_p \geq 2r(1+r)^{-2} + r^2 > 0$, and there exists a number $\theta_p$ and a number $t_p$ such that

$$|c_p| = \left(\frac{r}{1+r}\right)^2 \theta_p^2, \quad \Re c_p = \left(\frac{r}{1+r}\right)^2 \theta_p t_p,$$

$$0 \leq \theta_p \leq 1, -1 \leq t_p \leq 1.$$

Then,

$$0 \leq \frac{|c_p| - \Re c_p}{2\beta_p \beta_{p+1}} \leq \frac{(1+r)^2 \theta_p (1-t_p)}{2(2+r-r\theta_p t_p)(2+r-r\theta_{p-1} t_{p-1})}.$$

Since the derivative with respect to $r$ of the last expression is non-negative, it does not exceed

$$\frac{2\theta_p (1-t_p)}{(3-\theta_p t_p)(3-\theta_{p-1} t_{p-1})},$$

so that, if $h_p = (1-t_p)/2$ and $g_p = [1+\theta_p h_p]^{-1}$, we have:

$$|c_p| - \Re c_p \leq 2\beta_p \beta_{p+1} (1 - g_p) g_{p+1},$$

$$0 < g_p \leq 1, p = 1, 2, \cdots .$$

Thus, the even part of $f(a)$ is positive definite and, being bounded, is convergent.

George Copp has improved Theorem 4 by showing that $f(a)$ converges if there is a positive number $r$ not greater than 1 such that $|a_{2p}| \leq r^2$ and $|a_{2p-1}| \geq (1+r)^2, \ p = 1, 2, \cdots$. He also showed that $f(a)$ converges if there exists a positive number $r$ less than 1 such that, for each positive integer $p$, $|a_{2p}| \leq r^2$ and $|a_{2p-1}| \leq (1-r)^2$ or $|a_{2p-1}| \geq (1+r)^2$.

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7 AT, p. 69.
8 AT, p. 112.
9 George Copp, Some convergence regions for a continued fraction, Dissertation, The University of Texas, 1950.