ON THE LOCATION OF THE ZEROS OF CERTAIN ORTHOGONAL FUNCTIONS

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Let $R$ be a finite region of the $z$-plane and let $L^2(R)$ denote the class of functions $f(z)$ each analytic in $R$ with $\int_R \left| f(z) \right|^2 dS < \infty$. Let the points $\beta_1, \beta_2, \beta_3, \ldots$ be given interior to $R$ and let $\phi_n(z)$ be that function of class $L^2(R)$ for which $\phi_n(\beta_1) = \phi_n(\beta_2) = \cdots = \phi_n(\beta_{n-1}) = 0$, $\phi_n(\beta_n) = 1$, and which minimizes $\int_R \left| \phi_n(z) \right|^2 dS$ over the class $L^2(R)$. If the points $\beta_1, \beta_2, \beta_3, \ldots$ are not all distinct these requirements of interpolation on $\phi_n(z)$ are to be interpreted in the usual way in the theory of interpolation, to refer to the vanishing of suitable derivatives of $\phi_n(z)$ in multiple points $\beta_k$.

The purpose of this note is to establish results on the location of zeros of the functions $\phi_n(z)$. Our main result is the

**Theorem.** Let $R$ be a finite region of the $z$-plane which contains in its interior the points $\beta_1, \beta_2, \beta_3, \ldots$ and all limit points of the $\beta_n$. Then any circle which together with its interior lies in $R$ and which contains in its interior all limit points of the $\beta_n$, contains in its interior no zero of $\phi_n(z)$ (other than $\beta_k$, $k = 1, \ldots, n-1$) for $n$ sufficiently large.

The functions $\phi_n(z)$ were first introduced by Bergman [1] in the case that $\beta_n$ is independent of $n$, and were later studied by Walsh and Davis [6] in the case that the $\beta_n$ approach a limit. The totality of zeros of such functions were first studied by the present writers [2] in the analogous case that the double integral over $R$ is replaced by a line integral over the boundary of $R$; our theorem and other results on the totality of zeros of the $\phi_n(z)$ are of particular significance with reference to the asymptotic properties of the functions, and to the divergence of series $\sum_{n=1}^\infty a_n \phi_n(z)$.

The existence of the functions $\phi_n(z)$ follows readily from the theory of normal families, and uniqueness is also easily proved [2]. We introduce the normalized functions $\phi_n^*(z) = \phi_n(z) / \left( \int_R \left| \phi_n(z) \right|^2 dS \right)^{1/2}$.

A rapid sketch of the ideas underlying the proof is as follows. The functions $\phi_n^*(z)$ are bounded in norm in $R$, hence uniformly bounded in absolute value on any closed set interior to $R$. The fact that $\phi_n^*(z)$ has $n-1$ zeros in $R$ but not near the boundary of $R$ implies that
\( \phi_n(z) \to 0 \) uniformly on any closed set in \( R \). Thus the subset of \( R \) on which \( |\phi_n(z)| \) is not small lies near the boundary of \( R \), and the situation is similar to that in which the given functions are defined in terms of a line integral over the boundary of \( R \).

As a first step in the formal proof, we establish the

**Lemma.** Let \( \phi(z) \) be analytic and of modulus not exceeding the constant \( L \) for \( |z| \leq A \) and let \( \phi(\beta_k) = 0, \ k = n_0, n_0 + 1, \ldots, n - 1, \) with \( |\beta_k| \leq A\rho, \rho < 1. \) Then in \( |z| \leq |Z| < A \) we have

\[
|\phi(z)| \leq L \left( \frac{|Z/A| + \rho}{1 + \rho |Z/A|} \right)^{n-1-n_0}.
\]

If \( A = 1 \) we have by the maximum principle

\[
|\phi(z)| \leq \left| \frac{1}{n_{n_0}} \prod_{k=n_0}^{n-1} \frac{z - \beta_k}{1 - \beta_k z} \right| \leq L \quad \text{for} \quad |z| \leq 1,
\]

and since \([3, \text{p. 290}]\)

\[
\left| \frac{z - \beta_k}{1 - \beta_k z} \right| \leq \left| \frac{Z + \rho}{1 + \rho |Z|} \right| < 1, \quad \text{for} \quad |z| \leq |Z| < 1,
\]

the result follows for this case. The conclusion of the lemma for arbitrary \( A \) follows immediately by making the transformation \( z' = z/A \).

We now turn to the proof of the theorem. If \( S_1 \) and \( S_2 \) are any two point sets, \( d(S_1, S_2) \) shall denote the distance from \( S_1 \) to \( S_2 \). The closure of \( S_1 \) is denoted by \( \overline{S}_1 \). Let \( T \) be any circle (i.e. circumference) which together with its interior \( D \) lies in \( R \) such that the points \( \beta_k, k \geq n_0, \) lie in \( D \). Since \( d(T, C) > 0, \) where \( C \) is the boundary of \( R \), there exists a constant \( d \) such that \( d(\beta_n, T) \geq d > 0 \) when \( n \geq n_0 \). We now choose a circle \( T_1 \) contained in \( R \) and containing \( T \) in its interior; we consider the integral \( \iint_{R-D_1} \phi_n^*(z) |z^2(z-\alpha)^{-1}|dS, \) where \( D_1 \) is the interior of \( T_1 \) and \( \alpha \) is an arbitrary point in \( D \). We can interpret the conjugate of this integral as the force at \( z = \alpha \) due to a spread of non-negative matter over \( R - D_1 \) which repels according to the law of inverse distance. Since the set \( R - D_1 \) lies exterior to \( T \) and \( \alpha \) is interior to \( T \), this force is equal to the force at \( \alpha \) due to the same total mass concentrated at a suitable point \( \beta_n' \) exterior to \( T \) \([4, \text{pp. 13, 247}]\):

\[
(1) \quad \iint_{R-D_1} |\phi_n^*(z)|^2 \frac{dS}{z-\alpha} = \int_{R-D_1} |\phi_n^*(z)|^2 dS / (\beta_n' - \alpha).
\]
We shall now show that the assumption $\phi_n^*(\alpha) = 0$ ($\alpha \neq \beta_k$, for $k = 1, 2, \ldots, n$) implies that the point $\beta_n$ cannot lie exterior to $T$ when $n$ is sufficiently large, and we are thus led to a contradiction. If $\phi_n^*(\alpha) = 0$ the function $(z - \beta_n)\phi_n^*(z)/(z - \alpha)$ when suitably defined at $z = \alpha$ is of class $L^2(R)$ and vanishes in the points $\beta_1, \beta_2, \ldots, \beta_n$ and hence (compare [5]) is orthogonal to $\phi_n^*(z)$. That is to say,

$$
\int\int_R \frac{z - \beta_n}{z - \alpha} \left| \phi_n^*(z) \right|^2 dS = 0,
$$

(2) $\int\int_R \left| \phi_n^*(z) \right|^2 \frac{dS}{z - \alpha} = \int\int_R \left| \phi_n^*(z) \right|^2 dS / (\beta_n - \alpha) = \frac{1}{\beta_n - \alpha}$.

We next show that as a consequence of (1), (2), and the lemma we have for given $\eta (>0)$ and for all $n$ sufficiently large

(3) $\left| \frac{1}{\beta_n' - \alpha} \right| < \frac{2}{d(T, T_1)}$;

(4) $\left| \frac{1}{\beta_n - \alpha} - \frac{1}{\beta_n' - \alpha} \right| < \eta$.

Let $T_2$ be a circle concentric with $T_1$ which together with its interior lies in $R$ and which contains $T_1$ in its interior. Since $\int_{z=\alpha} \phi_n^*(z) \frac{dS}{z - \alpha} = 1$ there exists [3, p. 96] a constant $L$ depending on $T_2$ but not on $\phi_n^*(z)$ such that $\left| \phi_n^*(z) \right| \leq L$ for $z$ on and interior to $T_2$. We let $T_2$ play the rôle of the circle $|z| = A$ in the lemma. It then follows from the lemma that when $n > n_0 + 1$ there exists a positive constant $r (<1)$ independent of $n$ and of $z$ on $D_1$ such that

(5) $\left| \phi_n^*(z) \right| \leq L r^{n-1 - n_0}$, $z$ on $\overline{D_1}$.

The finite region $R$ is contained in a circle of radius $M$ and by (5) there exists an $n_1 (> n_0 + 1)$ such that when $n \geq n_1$ we have $\left| \phi_n^*(z) \right|^2 < 1/2\pi M^2$, $z$ in $\overline{D_1}$. Thus when $n \geq n_1$ we have

$$
\int\int_{R - \overline{D_1}} \left| \phi_n^*(z) \right|^2 dS - 1 = \int\int_{R - \overline{D_1}} \left| \phi_n^*(z) \right|^2 dS - \int\int_R \left| \phi_n^*(z) \right|^2 dS
$$

$$
= \int\int_{\overline{D_1}} \left| \phi_n^*(z) \right|^2 dS < \frac{1}{2},
$$

or

$$
\int\int_{R - \overline{D_1}} \left| \phi_n^*(z) \right|^2 dS > \frac{1}{2} \quad \text{when } n \geq n_1.
$$
Also
\[
\left| \iint_{R - D_1} \left| \frac{\phi_n^*(z)}{z - \alpha} \right|^2 \frac{dS}{z - \alpha} \right| \leq \frac{1}{d(T, T_1)} \iint_R \left| \frac{\phi_n^*(z)}{z - \alpha} \right|^2 dS = \frac{1}{d(T, T_1)}
\]
for all \( n \) and all \( \alpha \) interior to \( T \). Hence, from this last inequality and from (1)
\[
\left| \frac{1}{\beta_n' - \alpha} \right| = \left| \iint_{R - D_1} \left[ \left| \phi_n^*(z) \right|^2/(z - \alpha) \right] dS / \iint_{R - D_1} \left| \phi_n^*(z) \right|^2 dS \right| \leq \frac{2}{d(T, T_1)}, \quad n \geq n_1,
\]
and (3) is established.

By virtue of (5) there exists an \( n_2 (\geq n_0 + 1) \) such that when \( n \geq n_2 \)
\[
\left| \phi_n^*(z) \right| < \frac{\eta d(T, T_1)}{4\pi M^2}, \quad z \text{ in } \overline{D}_1.
\]
Thus, for \( z \) on \( T_1 \) and hence for \( z \) in \( D_1 \) we also have \( (n \geq n_2) \) by (6)
\[
\left| \frac{\phi_n^*(z)}{z - \alpha} \right| < \frac{\eta}{4\pi M^2},
\]
since the function \( \phi_n^*(z)/(z - \alpha) \) is analytic in \( \overline{D}_1 \) (when suitably defined for \( z = \alpha \)). Then, by (6) for \( n \geq n_2 \)
\[
1 - \iint_{R - D_1} \left| \phi_n^*(z) \right|^2 dS = \iint_R \left| \phi_n^*(z) \right|^2 dS - \iint_{R - \overline{D}_1} \left| \phi_n^*(z) \right|^2 dS = \iint_{D_1} \left| \phi_n^*(z) \right|^2 dS < \frac{\eta d(T, T_1)}{4},
\]
and by (7) for \( n \geq n_2 \)
\[
\left| \frac{1}{\beta_n - \alpha} - \iint_{R - \overline{D}_1} \left| \frac{\phi_n^*(z)}{z - \alpha} \right|^2 \frac{dS}{z - \alpha} \right|
\]
\[
= \left| \iint_{R} \phi_n^*(z) \left| \frac{dS}{z - \alpha} \right| - \iint_{R - \overline{D}_1} \phi_n^*(z) \left| \frac{dS}{z - \alpha} \right| \right|
\]
\[
= \iint_{D_1} \left| \frac{\phi_n^*(z)}{z - \alpha} \right|^2 < \frac{n}{4}.
\]
Hence, when \( n \geq \max (n_1, n_2) \) we have
We are now in a position to show that $\beta_n'$ cannot lie exterior to $T$. If in (4) we set $\eta = 1/d(T, T_1)$, it follows from (3) and (4) that for $n$ sufficiently large (and independent of $\alpha$ in $D$) we have $1/|\beta_n - \alpha| < 3/d(T, T_1)$. Since $1/2M < 1/|\beta_n - \alpha|$ for all $n$ sufficiently large and for all $\alpha$ in $D$, we have by setting $\eta = 1/4M$ in (4) that $1/4M < 1/|\beta_n - \alpha|$ for $n$ sufficiently large and for all $\alpha$ in $D$. Thus for all $n$ sufficiently large we have

$$
\frac{1}{4M} < \frac{1}{|\beta_n - \alpha|} < \frac{3}{d(T, T_1)}, \quad \frac{1}{4M} < \frac{1}{|\beta_n' - \alpha|} < \frac{3}{d(T, T_1)}.
$$

Since the function $f(Z) = 1/Z$ is uniformly continuous on any closed bounded set $B$ of the $Z$-plane not containing $Z = 0$, there corresponds to arbitrary $\epsilon (> 0)$ a $\delta (> 0)$ such that for all $Z_1$ and $Z_2$ on $B$ with $|Z_1 - Z_2| < \delta$ we have $|1/Z_1 - 1/Z_2| < \epsilon$. We now choose $B$ as the set $1/4M \leq |Z| \leq 3/d(T, T_1)$ with $\epsilon = \delta = d(\beta_n, T)$. In (4) we set $\eta = \delta$, whence it follows that there exists an $n_3$ such that when $n \geq n_3$ we have $|1/(\beta_n - \alpha) - 1/(\beta_n' - \alpha)| < \delta$ and such that (8) is valid. Then with $Z_1 = 1/(\beta_n - \alpha)$, $Z_2 = 1/(\beta_n' - \alpha)$ when $n \geq n_3$ we have $|\beta_n - \alpha| - (\beta_n' - \alpha)| < \delta \leq d(\beta_n, T)$ and hence the points $\beta_n'$ cannot lie exterior to $T$ when $n \geq n_3$. Since the choice of $n_3$ is independent of $\alpha$ interior to $T$, this contradiction completes the proof of the theorem.

The application of this theorem in the study of asymptotic properties of the $\phi_n^*(z)$ and in the study of divergence of series $\sum a_n \phi_n^*(z)$ is wholly analogous to the treatment previously given [2] and is left to the reader.

**Bibliography**


PARTIALLY BOUNDED CONTINUED FRACTIONS

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For each complex number sequence $a$, $f(a)$ denotes the continued fraction

$$
\frac{1}{a_1 + \frac{a_2}{1 + \frac{a_3}{1 + \ddots}}}
$$

The statement that $f(a)$ is partially bounded means that the sequence $a$ has a bounded infinite subsequence. If $f(a)$ is partially bounded, the series $\sum |b_p|$ diverges, where $b_1 = 1$, $a_p = 1/b_p b_{p+1}$, $p = 1, 2, \ldots$, a necessary condition for convergence of $f(a)$.

Any continued fraction $f(a)$ such that $\sum |b_p|$ diverges is convergent provided its even and odd parts are absolutely convergent, i.e. provided the series $\sum |f_{2p+2} - f_{2p}|$ and $\sum |f_{2p+1} - f_{2p+1}|$ are convergent, where $\{f_p\}_{p=1}^{\infty}$ is the sequence of approximants. The simple convergence of the even and odd parts of $f(a)$, together with the divergence of $\sum |b_p|$, is not sufficient for the convergence of $f(a)$, (Theorem 3). However, the simple convergence of the even and odd parts of the partially bounded continued fraction $f(a)$ is sufficient for the convergence of $f(a)$. In fact, we have this theorem:

**Theorem 1.** Suppose there is a positive integer $k$ such that the sub-

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1. $f(a)$ is called bounded if the sequence $a$ is bounded—a condition equivalent to the boundedness of a certain infinite matrix. Cf. H. S. Wall, Analytic theory of continued fractions, 1948, p. 110. (Referred to later on as AT.)