DETERMINANTS OF HARMONIC MATRICES

J. S. MACNERNEY

This paper is concerned with extensions of a theorem by H. S. Wall (Theorem 3 of [3]): if $M$ is a $2 \times 2$ harmonic matrix and $F$ corresponds to $M$ then $\det M = 1$ only in case $F_{11} = -F_{22}$.

As in [3] let $H_n$ denote the class of $n \times n$ harmonic matrices and $\Phi_n$ the class of $n \times n$ matrices $F$ of complex-valued functions from the real numbers, continuous and of bounded variation on every interval, such that $F(0) = 0$. In [3] Wall has shown that the Stieltjes integral equation,

$$M(s, t) = I + \int_t^s dF(u) \cdot M(u, t),$$

defines a one-to-one correspondence $M \sim F$ between $H_n$ and $\Phi_n$. In studying this correspondence in a more abstract setting, the present author [1] has obtained the continuous product (or "product integral") representation,

$$M(s, t) = \prod_{i=1}^n \{ I + dF \} \text{ when } M \sim F.$$

By using this representation, we now have the following extension of Wall's theorem cited above:

**Theorem 1.** If $M$ is in $H_n$ and $M \sim F$ then

$$\det M(s, t) = \text{Exp} \left( \sum_{i=1}^n [F_{pp}(t) - F_{pp}(s)] \right).$$

**Proof.** Let $f = \sum_{i} F_{pp}$, and $g$ be the function from the ordered real number pairs $\{ s, t \}$ defined by:

$$\det \{ I + F(t) - F(s) \} = 1 + f(t) - f(s) + g(s, t).$$

Let $J$ be a number interval, $s$ and $t$ numbers in $J$, and $b$ a positive number.

If $u$ and $v$ are real numbers then $g(u, v)$ is a sum of products of $n$ factors, of which at least two have the form $F_{ij}(v) - F_{ij}(u)$. Thus there exist a natural number $N$ and a sequence $\{ r_p, h_p, k_p \}^N$ such that if $u$ and $v$ are numbers in $J$ then

Presented to the Society, November 19, 1955; received by the editors October 17, 1955, and, in revised form, December 9, 1955.

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\[ g(u, v) = \sum_{1}^{N} r_p(u, v) [h_p(v) - h_p(u)][k_p(v) - k_p(u)], \]

where each \( r_p \) is a bounded function from \( J \times J \) to the numbers, each \( h_p \) is one of the \( F_{ij} \), and each \( k_p \) is one of the \( F_{ij} \). From the uniform continuity of the \( h_p \) on \( J \) and the bounded variation of the \( k_p \) on \( J \), it follows that there exists a positive number \( c \) such that if \( \{u_i\}_{1}^{n} \) is a monotone number sequence and \( u_0 = s \) and \( u_m = t \) and \( |u_i - u_{i-1}| < c \) for \( i = 1, \cdots, m \) then \( \sum_{i}^{m} |g(u_{i-1}, u_i)| < b \); hence,

\[
\left| \prod_{1}^{m} \{1 + f(u_i) - f(u_{i-1})\} - \det \prod_{1}^{m} \{I + F(u_i) - F(u_{i-1})\} \right| \\
\leq \sum_{1}^{m} |g(u_{i-1}, u_i)| \cdot \exp \left( \sum_{1}^{m} |f(u_i) - f(u_{i-1})| \right. \\
\left. + \sum_{1}^{m} |f(u_i) - f(u_{i-1}) + g(u_{i-1}, u_i)| \right) \\
\leq b \exp \left( b + 2 \sum_{1}^{m} \int_{t}^{t} \right. \left. |dF_{pp}| \right).
\]

Formula (3) is now apparent, since \( \prod_{1}^{m} \{1 + df\} = \exp (f[t] - f[s]) \).

Remark. The formula (3) is a generalization of the well-known exponential form of the Wronskian of a fundamental set of solutions for an \( n \)th order linear differential equation.

It seems natural to ask for a similar result in the case of quasi-harmonic matrices [2]—the statement that the \( n \times n \) matrix \( M \) is quasi-harmonic means that \( M \) is an \( n \times n \) matrix of complex-valued functions from the ordered pairs \( \{s, t\} \) of real numbers, which, for each \( t \), are of bounded variation in \( s \) on every interval and which are quasi-continuous in \( t \) for each \( s \), and that, for each ordered triple \( \{r, s, t\} \) of real numbers, \( M(r, s) \cdot M(s, t) = M(r, t) \) and

\[
M(s, s) = \frac{1}{2} \left[ M(s-, s) + M(s, s-) \right] \\
= \frac{1}{2} \left[ M(s, s+) + M(s+, s) \right] = I.
\]

Let \( QH_n \) denote the class of \( n \times n \) quasi-harmonic matrices and \( Q\Phi_n \) the class of \( n \times n \) matrices \( F \) of complex-valued functions from the real numbers, of bounded variation on every interval, such that

\[
[F(r) - F(r-)]^2 = [F(r+) - F(r)]^2 = F(0) = 0 \quad \text{for each} \; r.
\]
In [2] we have shown that (1), with mean integrals replacing the Stieltjes integrals used by Wall, defines a one-to-one correspondence $M \sim F$ between $QH_n$ and $Q\Phi_n$ which extends the correspondence established by Wall in [3] and which is also determined by (2).

**Theorem 2.** If $M$ is in $QH_n$ and $M \sim F$ and $G$ is the "continuous part" of $F$ then

$$\text{(6)} \quad \det M(s, t) = \exp \left( \sum_1^n \left[ G_{pp}(t) - G_{pp}(s) \right] \right).$$

**Proof.** By the "continuous part" of $F$ we mean (as in proof of Theorem 2.4 of [2]) an element $G$ of $\Phi_n$ such that, if $r_1, r_2, \cdots$ is a simple number sequence such that if $s$ is a number at which $F$ is not continuous then there is a natural number $k$ such that $r_k = s$, there is a sequence $F_1, F_2, \cdots$ of elements of $Q\Phi_n$ such that

(i) $F_1 = G$ and, if $j$ is a natural number and $[a, b]$ is a number interval which does not contain $r_j$, then $F_{j+1}(b) - F_{j+1}(a) = F_j(b) - F_j(a)$ and $F_{j+1}(r_j) - F_{j+1}(r_j-) = F(r_j) - F(r_j-)$ and $F_{j+1}(r_j+) - F_{j+1}(r_j) = F(r_j+) - F(r_j)$, and

(ii) $F_k(s) \rightarrow F(s)$ as $k \rightarrow \infty$ for each real number $s$.

Let $F_1, F_2, \cdots$ be such a sequence and $M_k(s, t) = \prod \{ I + dF_k \}$ for each natural number $k$.

If $A$ is an $n \times n$ matrix of complex numbers and $A^2 = 0$ then det $\{ I + A \} = 1$. This may easily be seen as follows: let $P$ be the function from the complex numbers defined by $P(z) = \det \{ I + zA \}$; now $P$ is a polynomial satisfying the identity $P(z)P(-z) = 1$, so that $P$ has no zero; hence $P$ is constant and its only value is $P(0)$, which is 1.

Thus we see that $\det M_k(s, t) = \det M_1(s, t)$ for $k = 1, 2, \cdots$. The formula (6) now follows from Theorem 2.5 of [2] and Theorem 1 of the present paper.

**Bibliography**


**University of North Carolina**