AN EQUIVALENCE RELATION FOR COMPACT HAUSDORFF VARIETIES

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The usual framework for Fréchet equivalence of functions entails a Peano space, a metric space, and the class of continuous functions from the Peano space to the metric space [1]. We show that an equivalence relation can be defined in a more general context entailing functions from a compact Hausdorff space to a set, which need not have a topology, provided the functions satisfy a certain compatibility condition. The class of continuous functions from a compact Hausdorff space to a topological space is compatible, if in the latter space every compact set is closed. Therefore it is meaningful to consider compact Hausdorff varieties, which of course include Peano varieties, in topological spaces considerably more general than metric spaces. The formulation is of such a nature that the equivalence of two functions can be tested without reference to any range space topology. For separated uniform spaces equivalence may be reexpressed in terms of the uniformity. Within the Fréchet framework this equivalence is identical to Fréchet equivalence.

As the basic structure we take a compact Hausdorff space X with the collection of closed sets $\mathfrak{J}$, and a class of functions $\mathfrak{F}$ each of which is defined on X and takes values in a set Y, such that the following compatibility condition is satisfied: for all f, $g \in \mathfrak{F}$ and all $E \subset X$, if $E \in \mathfrak{J}$ then $f^{-1}g(E) \in \mathfrak{J}$. For every function $f \in \mathfrak{F}$ we define a topology $\mathfrak{J}_f$ for its range in the following standard manner: for all $E \subset f(X)$, $E \in \mathfrak{J}_f$ if and only if $f^{-1}(E) \in \mathfrak{J}$.

THEOREM. *For f, $g \in \mathfrak{F}$ the following conditions hold*:

(i) *the function f is closed and continuous for the topology $\mathfrak{J}_f$ on $f(X)$*;

(ii) *the space $(f(X), \mathfrak{J}_f)$ is compact Hausdorff*;

(iii) *if $g(X) \subset f(X)$ then g is closed and continuous for $\mathfrak{J}_f$*;

(iv) *if $g(X) = f(X)$ then $\mathfrak{J}_g = \mathfrak{J}_f$.*

PROOF. Clearly f is $\mathfrak{J}_f$-continuous by the definition of $\mathfrak{J}_f$. If $E \in \mathfrak{J}$ then $f^{-1}f(E) \in \mathfrak{J}$ since f is self-compatible. Thus $f(E) \in \mathfrak{J}_f$ and so f is $\mathfrak{J}_f$-closed.

Since f is continuous and $\mathfrak{J}$ is a compact topology, $\mathfrak{J}_f$ is a compact topology. Now observe that $\mathfrak{J}_f$ is a Hausdorff topology if and only if

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for every two distinct points $x, y \in f(X)$ there exist $\mathfrak{J}_f$-closed sets E' and E'' such that $x \in E'$, $x \notin E''$, $y \in E''$, $y \notin E'$, and $E' \cup E'' = f(X)$. Let x and y be two distinct points in $f(X)$, then $f^{-1}(x)$ and $f^{-1}(y)$ are disjoint. The set $f^{-1}(x)$ is $\mathfrak{J}$-closed since there exists a point $u \in X$ such that $f(u) = x$, $\{u\}$ is $\mathfrak{J}$-closed, and f is self-compatible; similarly $f^{-1}(y)$ is $\mathfrak{J}$-closed. The space $(X, \mathfrak{J})$ is compact Hausdorff and so normal; consequently there exist $\mathfrak{J}$-closed sets F' and F'' such that $f^{-1}(x) \subset F'$, $f^{-1}(x) \cap F'' = 0$, $f^{-1}(y) \subset F''$, $f^{-1}(y) \cap F' = 0$, and $F' \cup F'' = X$. Then $x \in f(F')$, $x \notin f(F'')$, $y \in f(F'')$, $y \notin f(F')$, and $f(F') \cup f(F'') = f(X)$, where $f(F')$ and $f(F'')$ are $\mathfrak{J}_f$-closed since f is a closed function. Hence the above criterion for the Hausdorff property is satisfied.

Now assume that $g(X) \subset f(X)$. If $E \in \mathfrak{J}_f$ then $g^{-1}(E) = g^{-1}ff^{-1}(E) \in \mathfrak{J}$, since $f^{-1}(E) \in \mathfrak{J}$ and f and g are mutually compatible. If $E \in \mathfrak{J}$ then $g(E) = ff^{-1}g(E) \in \mathfrak{J}_f$, since $g(X) \subset f(X)$, $f^{-1}g(E) \in \mathfrak{J}$, and f is a closed function. Thus g is closed and continuous for $\mathfrak{J}_f$.

Suppose that $g(X) = f(X)$. If $E \in \mathfrak{J}_f$ then $E = gg^{-1}(E) \in \mathfrak{J}_g$, since g is $\mathfrak{J}_f$-continuous and $\mathfrak{J}_g$-closed. Similarly we can show that $\mathfrak{J}_g \subset \mathfrak{J}_f$.

Because $(X, \mathfrak{J})$ is a compact Hausdorff space there is one and only one associated uniformity $\mathfrak{U}$; see [2]. The uniformity $\mathfrak{U}$ is the collection of all relations in $X \times X$ which are neighborhoods of the identity relation in the Cartesian product topology. This collection satisfies the usual uniformity conditions: each $U \in \mathfrak{U}$ contains the identity relation in $X \times X$; if $U \in \mathfrak{U}$ then $U^{-1} \in \mathfrak{U}$; if $U \in \mathfrak{U}$ then there is a $V \in \mathfrak{U}$ such that[1] $VV \subset U$; if $U, V \in \mathfrak{U}$ then $U \cap V \in \mathfrak{U}$; and if $U \in \mathfrak{U}$ and $U \subset V \subset X \times X$ then $V \in \mathfrak{U}$. The connection between the topology $\mathfrak{J}$ and the uniformity $\mathfrak{U}$ is the following: the $\mathfrak{J}$-open sets are all sets E for which for each $x \in E$ there is a $U \in \mathfrak{U}$ such that[2] $U(x) \subset E$. Similarly, every generated compact Hausdorff space $(f(X), \mathfrak{J}_f)$ has uniquely such an associated uniformity $\mathfrak{U}_f$. We now prove two useful connections between the uniformities $\mathfrak{U}$ and $\mathfrak{U}_f$.

THEOREM. *For $f \in \mathfrak{F}$ let $(f(X), \mathfrak{J}_f)$ be the space with uniformity $\mathfrak{U}_f$ that is generated by the function f from the compact Hausdorff space $(X, \mathfrak{J})$ with uniformity $\mathfrak{U}$. Then:*

(i) *for every $U \in \mathfrak{U}$ and $x \in f(X)$ the set $fUf^{-1}(x)$ is a $\mathfrak{J}_f$-neighborhood of x;*

(ii) *the collection of relations $\{W_f(U): U \in \mathfrak{U}\}$ in $f(X) \times f(X)$, where $W_f(U) = fUf^{-1}fUf^{-1}$, is a base for the uniformity $\mathfrak{U}_f$.*

PROOF. Assume that there is a $U \in \mathfrak{U}$ and an $x \in f(X)$ such that

[1] Two relations written in juxtaposition, such as "VV", will denote relational composition.

[2] For a relation U, $U(x)$ is the set of all y such that $(x, y) \in U$.

$fUf^{-1}(x)$ is not a $\mathfrak{J}_f$-neighborhood of x. Then there exists a syntax $\langle s, \mathfrak{N}\rangle$ in $f(X)$ which converges in the topology $\mathfrak{J}_f$ of $f(X)$ to x and which is disjoint from $fUf^{-1}(x)$; see [3]. In the space X there is a syntax $\langle t, \mathfrak{N}\rangle$ such that $\langle ft, \mathfrak{N}\rangle = \langle s, \mathfrak{N}\rangle$. Since $(X, \mathfrak{J})$ is compact there is a sub-syntax $\langle t, \mathfrak{M}\rangle$ of $\langle t, \mathfrak{N}\rangle$ which converges in the topology $\mathfrak{J}$ to some point $u \in X$. By the continuity of f the syntax $\langle ft, \mathfrak{M}\rangle$ converges in the topology $\mathfrak{J}_f$ to $f(u)$. But $\langle ft, \mathfrak{M}\rangle$ is a sub-syntax of $\langle s, \mathfrak{N}\rangle$ and so also converges to x. Since $\mathfrak{J}_f$ is a Hausdorff topology we have $f(u) = x$. Since $U(u)$ is a $\mathfrak{J}$-neighborhood of u, $\langle t, \mathfrak{M}\rangle$ is eventually in $U(u)$ and so eventually in $Uf^{-1}(x)$. Thus $\langle ft, \mathfrak{M}\rangle$ is eventually in $fUf^{-1}(x)$, and so there is a point of $\langle ft, \mathfrak{N}\rangle = \langle s, \mathfrak{N}\rangle$ in $fUf^{-1}(x)$. This is the desired contradiction.

For the second part of the theorem it is sufficient to show that for every $V \in \mathfrak{U}_f$ there is a $U \in \mathfrak{U}$ such that $W_f(U) \subset V$ and that for every $U \in \mathfrak{U}$ there is a $V \in \mathfrak{U}_f$ such that $V \subset W_f(U)$. Since f is continuous and $(X, \mathfrak{J})$ is compact, f is uniformly continuous. Thus, for every $V \in \mathfrak{U}_f$ there is a $U \in \mathfrak{U}$ such that $fU \subset Vf$, and so $fUf^{-1} \subset V$. For $V \in \mathfrak{U}_f$ we choose a $V' \in \mathfrak{U}_f$ such that $V'V' \subset V$, then a $U \in \mathfrak{U}$ such that $fUf^{-1} \subset V'$. Then $W_f(U) = fUf^{-1}fUf^{-1} \subset V'V' \subset V$.

By the first part $fUf^{-1}(x)$ is a neighborhood of x, for every $x \in f(X)$. Define $E(x)$ to be the interior of $fUf^{-1}(x)$, then $\{E(x): x \in f(X)\}$ is an open covering of $f(X)$. Since $\mathfrak{J}_f$ is a compact topology there is a $V \in \mathfrak{U}_f$, which can be taken as symmetric, such that $\{V(x): x \in f(X)\}$ is a refinement of $\{E(x): x \in f(X)\}$. Consequently, for every $y \in f(X)$ there exists an $x \in f(X)$ such that $V(y) \subset E(x) \subset fUf^{-1}(x)$, and so also $V(y) \times V(y) \subset fUf^{-1}(x) \times fUf^{-1}(x)$. From this we infer that $\bigcup\{V(y) \times V(y): y \in f(X)\}$ is contained in $\bigcup\{fUf^{-1}(x) \times fUf^{-1}(x): x \in f(X)\}$. If an $R \subset f(X) \times f(X)$ is symmetric then $RR = \bigcup\{R(x) \times R(x): x \in f(X)\}$. Assuming U to be symmetric, and so also fUf^{-1}, we have $VV \subset fUf^{-1}fUf^{-1} = W_f(U)$. But $V \subset VV$, and so for every symmetric $U \in \mathfrak{U}$ there is a $V \in \mathfrak{U}_f$ such that $V \subset W_f(U)$. Finally, the symmetry restriction may be dropped because every $U' \in \mathfrak{U}$ contains a symmetric $U \in \mathfrak{U}$.

Remark. The simpler and more obvious collection of relations $\{fUf^{-1}: U \in \mathfrak{U}\}$ cannot serve as a basis for $\mathfrak{U}_f$ even in the most elementary of situations. For example, let f be the function mapping the unit interval with the usual metric onto itself in the following manner: $f(x) = 3x/2$ for $0 \leqq x \leqq 1/3$; $f(x) = 1/2$ for $1/3 \leqq x \leqq 2/3$; and $f(x) = (3x-1)/2$ for $2/3 \leqq x \leqq 1$. Choose $U \in \mathfrak{U}$ so that each $U(y)$ is the sphere of radius $1/12$ about y. Then fUf^{-1} is not a neighborhood of the identity relation in the Cartesian product topology, and so cannot be an element of any base for the uniformity.

With the appropriate features of the basic structure now developed,

the relation $\sim$ for compact Hausdorff varieties can be defined.

DEFINITION. For $f, g \in \mathfrak{F}$, $f \sim g$ if and only if for every $U \in \mathfrak{U}$ there exists a homeomorphism $h: X \to X$ such that $fh \subset W_g(U)g$.

THEOREM. *For the relation $\sim$ the following conditions hold*:

(i) *for $f, g \in \mathfrak{F}$, $f \sim g$ if and only if for every $U \in \mathfrak{U}$ there exists a homeomorphism $h: X \to X$ such that $gh \subset W_g(U)f$*;

(ii) *for $f, g \in \mathfrak{F}$, if $f \sim g$ then $f(X) = g(X)$*;

(iii) *the relation $\sim$ is an equivalence relation.*

PROOF. We first observe that the definition of $\sim$ is equivalent to the analogous one involving only symmetric relations $U \in \mathfrak{U}$. Therefore we may assume all $U \in \mathfrak{U}$ that are employed to be symmetric. Assume that $f \sim g$, so that for every $U \in \mathfrak{U}$ there is a homeomorphism $h: X \to X$ such that $fh \subset W_g(U)g$. Then $fhg^{-1} \subset W_g(U)$, and by taking inverses $gh^{-1}f^{-1} \subset W_g(U)^{-1} = W_g(U^{-1}) = W_g(U)$. Hence $gh^{-1} \subset gh^{-1}f^{-1}f \subset W_g(U)f$, and h^{-1} serves as the required homeomorphism. Conversely, assume that for every $U \in \mathfrak{U}$ there is a homeomorphism $h: X \to X$ such that $gh \subset W_g(U)f$. Then $ghf^{-1} \subset W_g(U)$, $fh^{-1}g^{-1} \subset W_g(U)$, $fh^{-1} \subset fh^{-1}g^{-1}g \subset W_g(U)g$, and h^{-1} serves as the required homeomorphism.

Let $f \sim g$, so that for every $U \in \mathfrak{U}$ there is a homeomorphism $h: X \to X$ such that $fh \subset W_g(U)g$. Thus $f \subset W_g(U)gh^{-1}$, and so for every $x \in X$, $f(x) \in W_g(U)gh^{-1}(x) \subset W_g(U)g(X)$. But $g^{-1}g(X) = X$ and $U(X) = X$, therefore $f(x) \in g(X)$ for all $x \in X$. Since $f(X) \subset g(X)$ the function f is closed and continuous for the space $(g(X), \mathfrak{I}_g)$, for which a base for the uniformity $\mathfrak{U}_g$ is $\{W_g(U): U \in \mathfrak{U}\}$. By the result of the above paragraph $f \sim g$ implies that for every $U \in \mathfrak{U}$ there is a homeomorphism $h: X \to X$ such that $gh \subset W_g(U)f$. Then $g \subset W_g(U)fh^{-1}$, and $g(x) \in W_g(U)fh^{-1}(x) \subset W_g(U)f(X)$ for all $x \in X$. Because this holds for every $U \in \mathfrak{U}$ it is equivalent to asserting that $g(x) \in Vf(X)$ for every $V \in \mathfrak{U}_g$. So for every $V \in \mathfrak{U}_g$ there is a $y \in f(X)$ such that $g(x) \in V(y)$, or $y \in V^{-1}g(x)$, and so $g(x)$ is a $\mathfrak{I}_g$-limit point of $f(X)$. On the other hand, $f(X)$ is a $\mathfrak{I}_g$-closed set, which implies that $g(x) \in f(X)$.

Lastly, we show that $\sim$ is an equivalence relation. Clearly for every $U \in \mathfrak{U}$, $fi = f \subset W_f(U)f$ and the identity function i on X serves as the homeomorphism sufficient to guarantee $f \sim f$. Assume that $f \sim g$, then by the above result $f(X) = g(X)$ and both of the collections $\{W_f(U): U \in \mathfrak{U}\}$ and $\{W_g(U): U \in \mathfrak{U}\}$ are bases for the common uniformity $\mathfrak{U}_f = \mathfrak{U}_g$ of the range space. Thus for $U \in \mathfrak{U}$ there is a $V \in \mathfrak{U}$ such that $W_g(V) \subset W_f(U)$. By the result of the first paragraph for a given $V \in \mathfrak{U}$ there is a homeomorphism $h: X \to X$ such that $gh \subset W_g(V)f$, hence $gh \subset W_f(U)f$. We conclude that $g \sim f$. Assume that $e \sim f$ and $f \sim g$. Then $e(X) = f(X) = g(X)$ and $\{W_f(U): U \in \mathfrak{U}\}$ and $\{W_g(U):$

$U \in \mathfrak{U}\}$ is each of them a base for the uniformity of the common range space. Therefore for a given $U \in \mathfrak{U}$ there exist U', $U'' \in \mathfrak{U}$ such that $W_f(U')W_g(U'') \subset W_g(U)$. There also exist homeomorphisms h' and h'' such that $eh' \subset W_f(U')f$ and $fh'' \subset W_g(U'')g$. Thus $eh'h'' \subset W_f(U')fh'' \subset W_f(U')W_g(U'')g \subset W_g(U)g$. We conclude that $e \sim g$.

So far it has been assumed abstractly that the class of functions $\mathfrak{F}$ satisfies the compatibility condition, and no reference has been made to a topology on the set Y. If Y is topologized the natural functions to consider are the continuous functions. As the following theorem shows, the above results are then applicable under a mild restriction on the topology on Y, and the concept of compact Hausdorff varieties in topological spaces is indeed meaningful.

THEOREM. *Let $\mathfrak{F}$ be the class of continuous functions from the compact Hausdorff space $(X, \mathfrak{J})$ to a topological space $(Y, \mathcal{S})$ in which every compact set is closed. Then:*

(i) *the class of functions $\mathfrak{F}$ satisfies the compatibility condition;*

(ii) *for every $f \in \mathfrak{F}$ the topology $\mathfrak{J}_f$ on $f(X)$ agrees with the relative topology $\mathcal{S}_r$ obtained from $\mathcal{S}$.*

PROOF. Let f, $g \in \mathfrak{F}$ and let E be a $\mathfrak{J}$-closed set of X. Then E is compact and so $f(E)$ is compact in the topology $\mathcal{S}$. By the assumption for $(Y, \mathcal{S})$ the set $f(E)$ is $\mathcal{S}$-closed. Then $g^{-1}f(E)$ is $\mathfrak{J}$-closed.

Let E be a set which is closed in the topology $\mathcal{S}_r$ on $f(X)$. Then $E = f(X) \cap E'$, where E' is $\mathcal{S}$-closed. So $f^{-1}(E) = f^{-1}(f(X) \cap E') = f^{-1}(E')$ is $\mathfrak{J}$-closed, since f is $\mathcal{S}$-continuous. Since f is a closed function for the topology $\mathfrak{J}_f$ on $f(X)$ it follows that $E = ff^{-1}(E)$ is $\mathfrak{J}_f$-closed. Let E be a $\mathfrak{J}_f$-closed set, then E is compact in the $\mathfrak{J}_f$-topology. But $\mathcal{S}_r \subset \mathfrak{J}_f$, and so E is compact in the $\mathcal{S}_r$-topology, compact in the topology $\mathcal{S}$, closed therefore in the $\mathcal{S}$-topology, and so $\mathcal{S}_r$-closed.

Turning toward the Fréchet formulation, we obtain for separated uniform spaces an expression of functional equivalence in which the uniform structure of the range space enters explicitly.

THEOREM. *Let $\mathfrak{F}$ be the class of continuous functions from the compact Hausdorff space $(X, \mathfrak{J})$ with uniformity $\mathfrak{U}$ to a separated uniform space $(Y, \mathcal{S})$ with uniformity $\mathfrak{V}$. For $f, g \in \mathfrak{F}$, $f \sim g$ if and only if for every $V \in \mathfrak{V}$ there exists a homeomorphism $h: X \to X$ such that $fh \subset Vg$.*

PROOF. Since in a uniform space the separation property is equivalent to every compact set being closed, the class $\mathfrak{F}$ is compatible. Moreover, for any $g \in \mathfrak{F}$, the subset $g(X) \subset Y$ has a relativized uniformity $\mathfrak{V}_r$ consisting of the collection of relations of the form $V \cap (g(X) \times g(X))$, where $V \in \mathfrak{V}$. The topology obtained from $\mathfrak{V}_r$ is identical to the topology $\mathcal{S}_r$ obtained by relativizing $\mathcal{S}$ to $g(X)$. By the second

part of the theorem above $\mathfrak{J}_g$ is identical to $\mathcal{S}_r$, and in addition is compact; therefore $\{W_g(U): U\in\mathfrak{U}\}$ is a base for $\mathfrak{V}_r$. Now assume for $f, g\in\mathfrak{F}$ that $f\sim g$, so that for every $U\in\mathfrak{U}$ there is an $h: X\to X$ such that $fh\subset W_g(U)g$. For a given $V\in\mathfrak{V}$ there is a $U\in\mathfrak{U}$ such that $W_g(U)\subset V\cap(g(X)\times g(X))\subset V$, and so also $W_g(U)g\subset Vg$; consequently an $h: X\to X$ exists such that $fh\subset W_g(U)g\subset Vg$. Conversely, assume that for every $V\in\mathfrak{V}$ there is an $h: X\to X$ such that $fh\subset Vg$. Then $f\subset Vgh^{-1}$, implying $f(x)\in Vgh^{-1}(x)\subset Vg(X)$, for every $x\in X$ and $V\in\mathfrak{V}$. Equivalently, for any fixed $x\in X$ and every $V\in\mathfrak{V}$ there is a $y\in g(X)$ such that $y\in Vf(x)$, and so $f(x)$ is a limit point of $g(X)$. But $g(X)$ is $\mathfrak{J}_g$-compact, therefore $\mathcal{S}_r$-compact, and so $\mathcal{S}$-closed since the uniformity $\mathfrak{V}$ is assumed to be separated. This implies that $f(x)\in g(X)$, for all $x\in X$. Now assume for $x\in X$ that $z\in Vg(x)\cap g(X)$, then $(g(x), z)\in V$ and $z\in g(X)$. This together with $g(x)\in g(X)$ implies $(g(x), z)\in V\cap(g(X)\times g(X))$, or $z\in[V\cap(g(X)\times g(X))]g(x)$. Thus $Vg(x)\cap g(X)\subset[V\cap(g(X)\times g(X))]g(x)$, for all $x\in X$. Finally, for a choice of $U\in\mathfrak{U}$ there is a $V\in\mathfrak{V}$ such that $W_g(U)=V\cap(g(X)\times g(X))$, and so $W_g(U)g(x)=[V\cap(g(X)\times g(X))]g(x)$, for all $x\in X$. For this $V\in\mathfrak{V}$ there is an $h: X\to X$ such that $fh\subset Vg$, hence $fh(x)\in Vg(x)$ for all $x\in X$. But $fh(x)\in g(X)$ for all $x\in X$, and so $fh(x)\in Vg(x)\cap g(X)\subset[V\cap(g(X)\times g(X))]g(x)=W_g(U)g(x)$ for all $x\in X$. We conclude that $f\sim g$.

As an immediate corollary to this theorem we prove our final result, which involves Fréchet equivalence.

COROLLARY. *Let $\mathfrak{F}$ be the class of continuous functions from a Peano space to a metric space. For $f, g\in\mathfrak{F}$, $f\sim g$ if and only if f and g are Fréchet equivalent.*

PROOF. Since the Peano space is compact Hausdorff and the metric space is a separated uniform space the above theorem is applicable upon observing the following: the class of relations $\{V_d: d>0\}$ in the Cartesian product, where, for every x, $V_d(x)$ is the sphere of radius d about x, is a base for the metric uniformity; two continuous functions f and g are Fréchet equivalent if and only if for every $d>0$ there exists an $h: X\to X$ such that $fh(x)\in V_dg(x)$ for all x.

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