A NOTE ON SECOND-ORDER NONLINEAR
DIFFERENTIAL EQUATIONS

W. R. UTZ

In this note we prove a theorem on the behavior, as \( t \to \infty \), of solutions of the nonlinear equation

\[
x'' + \alpha(x)x' + \beta(x)x = 0
\]

and prove a corollary to this theorem concerning the behavior of the solutions, as \( t \to \infty \), of

\[
x'' + g(x') + cx = 0.
\]

It will be convenient to be able to refer to the following elementary and intuitively obvious lemma in both proofs.

**Lemma.** Suppose that \( x(t) \) is a real function for which \( x''(t) \) is defined for all \( t \geq a \). (i) If for all \( t \geq a \), \( x'(t) < 0 \) and \( x''(t) \neq 0 \), then \( \lim_{t \to \infty} x(t) = -\infty \). (ii) If for all \( t \geq a \), \( x'(t) > 0 \) and \( x''(t) \neq 0 \), then \( \lim_{t \to \infty} x(t) = \infty \).

Throughout the note a function is said to oscillate or be oscillatory when and only when it has arbitrarily large zeros.

**Theorem.** If \( \alpha(x) \) and \( \beta(x) \) are real functions such that for all real \( x \),

\[
\alpha(x) \leq 0, \quad \beta(x) > 0
\]

and if \( x(t) \) is a solution of (1) valid for all large \( t \), then \( x(t) \) oscillates or, for all large \( t \), \( x(t) \) is monotone. In case \( x(t) \) is monotone increasing, \( \lim_{t \to \infty} x(t) > 0 \) and in case \( x(t) \) is monotone decreasing, \( \lim_{t \to \infty} x(t) < 0 \).

**Proof.** Suppose that \( x(t) \) does not oscillate. Then for large \( t \), \( x \) is of fixed sign. We assume that \( x > 0 \) and note that a parallel argument holds for \( x < 0 \). If \( x'(t) = 0 \), then \( x'' = -\beta(x)x \) and \( x'' < 0 \) hence \( x'(t) \) cannot have arbitrarily large zeros as \( x(t) \) would have infinitely many critical values all of which would be maxima. Thus, for \( t \) large, \( x'(t) \) is of fixed sign.

**Case 1.** If \( x' < 0 \) then, since

\[
x'' = -\alpha(x)x' - \beta(x)x,
\]

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$x'' < 0$ and, by the lemma, $\lim_{t \to \infty} x(t) = -\infty$ which contradicts $x(t) > 0$ for large $t$.

Case 2. If $x' > 0$ then $x(t)$ is a montone increasing function and since $x > 0$, $\lim_{t \to \infty} x(t) > 0$.

Corollary. Let $c$ be a positive constant and suppose that $g(0) = 0$ and that for all real $z$, $g'(z) \leq 0$. If $x(t)$ is a solution of (2) valid for all large $t$, then $x(t)$ is oscillatory, $\lim_{t \to \infty} x(t) = \infty$, or $\lim_{t \to \infty} x(t) = -\infty$.

Proof. If we set $v = x'$, then

(3) $v'' + g'(v)v' + cv = 0$

which is (1) with

$$g'(v) = a(v) \leq 0, \quad c = \beta(v) > 0$$

hence our theorem applies to (3).

If $v = 0$, then $x(t) = 0$ since $g(0) = 0$. Hence $x$ is oscillatory.

Suppose that $v$ is oscillatory but $x$ is not oscillatory. Then $v$ has arbitrarily large zeros and $x$ is eventually of fixed sign. When $v = 0$, $x'' = -cx$ and $x$ has extrema for arbitrarily large $t$ which are all maxima or all minima. As this is impossible we conclude that $v$ oscillatory implies $x$ oscillatory.

According to our theorem, if $v$ is not oscillatory, then $v$ is monotone. Assume $v$ is monotone increasing, then, according to the theorem above, eventually $v > 0$. Thus for sufficiently large $t$, $x' > 0$, $x'' \geq 0$ and by the lemma $\lim_{t \to \infty} x(t) = \infty$.

Similarly, if $v$ is monotone decreasing, $\lim_{t \to \infty} x(t) = -\infty$. This completes the proof of the corollary.

Equations (1) and (2) may be considered as generalizations, in two directions, of the linear equation

(4) $x'' + dx' + ex = 0$

with constant coefficients $d \leq 0$, $e > 0$. Thus we have shown that, for large $t$, solutions of (1) and (2), under the hypotheses of the theorem and corollary, behave as the solutions of (4).

University of Missouri and Institute for Advanced Study