A PROPERTY OF ORDERED RINGS

A. A. ALBERT

In this note we shall provide the last essential element in a simple proof of the theorem of Wagner [1] which states that every ordered ring satisfying a nontrivial polynomial identity is commutative.

It is clear that an ordered ring $A$ has no divisors of zero. Moreover, the existence of a nontrivial identity is easily seen to imply that if $a \neq 0$ and $b \neq 0$ are in $A$ there exist elements $c \neq 0$ and $d \neq 0$ in $A$ such that

\[(1) \quad ac = bd.\]

Hence $A$ is what O. Ore [2] has called a regular ring. Then $A$ can be imbedded in a unique quotient ring $B$. Every element of $B$ can be expressed as a product

\[(2) \quad a = b^{-1}a,\]

for $b \neq 0$ and $a$ in $A$. By (1) we can always write

\[(3) \quad a = b^{-1}a = dc^{-1}.\]

Since $\alpha = (-b)^{-1}(-a) = (-d)(-c)^{-1}$ we can assume that, in the case where $A$ is ordered, the denominators $b$ and $c$ are always positive. We now derive the following sequence of simple lemmas.

**Lemma 1.** Let $\alpha = b^{-1}a = dc^{-1}$ where $b > 0$, $c > 0$. Then $a$ and $d$ have the same sign.

For $b^{-1}a = dc^{-1}$ if and only if $bd = ac$. Since $b > 0$ and $c > 0$ the elements $bd$ and $ac$ of $A$ can be equal only if $a$ and $d$ have the same sign.

**Lemma 2.** Let $\alpha = b^{-1}a = c^{-1}f$ where $b > 0$, $c > 0$. Then $a$ and $f$ have the same sign.

For we use (3) to write $\alpha = dc^{-1}$ where $c > 0$, and $d$ has the same sign as $a$. By Lemma 1 we know that $f$ has the same sign as $d$ and hence the same sign as $a$.

Since the sign of $a$ is unique we may say that $\alpha = b^{-1}a > 0$ if $a > 0$, $\alpha < 0$ if $a < 0$, $\alpha = 0$ if $a = 0$. We may then prove the following result.

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Lemma 3. Let $\alpha \neq 0$ be in $B$ and let there exist positive elements $a$ and $b$ of $A$ such that $c = ab$ is in $A$. Then $c$ and $\alpha$ have the same sign.

For $a\alpha = cb^{-1}$ has the same sign as $c$ by our definition of sign. Also $cb^{-1} = e^{-1}f$ where $e > 0$ and $f$ has the same sign as $c$. Then $\alpha = (ea)^{-1}f$, $ea > 0$, $\alpha$ has the same sign as $f$ and hence the same sign as $c$.

Lemma 4. Let $\alpha$ and $\beta$ be in $B$ and $\alpha > 0$, $\beta > 0$. Then $\alpha + \beta$ and $\alpha \beta$ are positive.

For we may write $\alpha = a^{-1}b$, $\beta = dc^{-1}$ with $a$, $b$, $c$, $d$ all positive. Then $a(\alpha + \beta)c = a(a^{-1}b + dc^{-1}) = bc + ad > 0$. By Lemma 3 we have $\alpha + \beta > 0$. Also $a(\alpha \beta)c = (a\alpha)(\beta c) = bd > 0$ and so $\alpha \beta > 0$.

If $\alpha < 0$ and $\beta < 0$ then $-\alpha > 0$, $-\beta > 0$, $(-\alpha)(-\beta) = \alpha \beta > 0$. Similarly if $\alpha < 0$ and $\beta > 0$ we have $-(\alpha \beta) = (-\alpha)\beta > 0$ and $\alpha \beta < 0$. We have completed a proof of the following result.

Theorem. The quotient ring of an ordered regular ring is ordered.

As a consequence of results of Amitsur [3] and Kaplansky [4] we have the property which states that if an ordered ring $A$ satisfies a nontrivial polynomial identity the quotient ring $B$ also satisfies the identity and is finite-dimensional over its center $F$. By our theorem $B$ is ordered and this order clearly implies that $F$ is ordered. But then it is known [5] that $B$ is commutative and so we have Wagner's result that $A$ is commutative.

References


The University of Chicago