DERIVATIONS OF NILPOTENT LIE ALGEBRAS

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In a recent note Jacobson proved [1] that, over a field of characteristic 0, a Lie algebra with a nonsingular derivation is nilpotent. He also noted that the validity of the converse was an open question. The purpose of this note is to supply a strongly negative answer to that question and to point out some of the immediate problems which this answer raises.

Suppose then that \( \Phi \) is a field of characteristic 0 and that \( \mathcal{L} \) is the 8 dimensional algebra over \( \Phi \) described in terms of a basis \( e_1, e_2, \ldots, e_8 \) by the following multiplication table:

1. \( [e_1, e_2] = e_6, \)
2. \( [e_1, e_3] = e_6, \)
3. \( [e_1, e_4] = e_7, \)
4. \( [e_1, e_5] = -e_8, \)
5. \( [e_1, e_6] = e_8, \)
6. \( [e_2, e_4] = e_6, \)
7. \( [e_2, e_5] = e_7, \)
8. \( [e_2, e_6] = -e_7, \)
9. \( [e_2, e_7] = -e_8, \)
10. \( [e_2, e_8] = e_8. \)

In addition \( [e_i, e_j] = -[e_j, e_i] \) and for \( i < j \) \( [e_i, e_j] = 0 \) if it is not in the table above. Note that all triple products \( [[[e_i e_j] e_k]] \) vanish if one index is >4. It is convenient to use a symmetry in the table above. Denote by \( A \) the linear transformation induced in \( \mathcal{L} \) by the mapping

\[
\begin{pmatrix}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\
e_3 & e_4 & e_1 & e_2 & -e_5 & -e_6 & -e_8 & -e_7
\end{pmatrix}
\]

A direct check shows that \( A \) is an automorphism of \( \mathcal{L} \). By observing

\[
[[e_1 e_2] e_3] + [[e_2 e_3] e_1] + [[e_3 e_1] e_2] = e_7 - e_7 = 0
\]

and

\[
[[e_1 e_2] e_4] + [[e_2 e_4] e_1] + [[e_4 e_1] e_2] = 0,
\]

and by applying \( A \) to each we conclude that \( \mathcal{L} \) is a Lie algebra.

Since \( \mathcal{L}^2 = \{e_6, e_8, e_7, e_8\} \), \( \mathcal{L}^3 = \{e_7, e_8\} \), \( \mathcal{L}^4 = \{0\} \), \( \mathcal{L} \) is nilpotent.

**Theorem.** If \( D \) is a derivation of \( \mathcal{L} \) then \( \mathcal{L} D \subset \mathcal{L}^2 \); hence every derivation is nilpotent.

**Proof.** Suppose \( e_i D = \sum \delta_{ij} e_j \), \( 1 \leq i \leq 8, \quad 1 \leq j \leq 8 \). The equations

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\[ [e_1, e_2]D = e_5D = \delta_{56}e_6 + \delta_{68}e_8 + \delta_{67}e_7 + \delta_{58}e_8, \]
\[ [e_1D, e_2] = \delta_{11}e_6 - \delta_{14}e_8 + \delta_{16}e_7 - \delta_{13}e_6, \]
\[ [e_1, e_2D] = \delta_{22}e_6 + \delta_{23}e_8 + \delta_{24}e_7 - \delta_{26}e_8, \]

implies
\[ \delta_{65} = \delta_{11} + \delta_{22}, \quad \delta_{56} = \delta_{23} - \delta_{14}, \quad \delta_{67} = \delta_{16} + \delta_{24}, \quad \delta_{58} = -\delta_{13} - \delta_{26}. \]

With the observation that for \( i \leq 4 \) and \( 5 \leq k \leq 8 \) there is exactly one \( j \leq 6 \) for which \([e_i, e_j] = \pm e_k\) it follows from (2) that
\[ \delta_{65} = \delta_{14} + \delta_{32}, \quad \delta_{66} = \delta_{11} + \delta_{33}, \quad \delta_{67} = \delta_{16} + \delta_{34}, \quad \delta_{68} = \delta_{12} - \delta_{35}, \]
from (3) that
\[ \delta_{77} = \delta_{11} + \delta_{44}, \quad \delta_{78} = \delta_{16} - \delta_{46}, \quad \delta_{13} = \delta_{42}, \quad \delta_{12} = -\delta_{43} \]
and from (4) that
\[ \delta_{57} = \delta_{13}, \quad \delta_{88} = \delta_{11} + \delta_{56}. \]

The automorphism \( A \) transforms \( D \) into a derivation \( D^* \) by \( A^{-1}DA = D^* \) and this implies that with each equation \( \delta_{ij} = \delta_{kl} + \delta_{mn} \) there is also valid \( \delta_{\alpha(i), \alpha(j)} = \delta_{\alpha(k), \alpha(l)} + \delta_{\alpha(m), \alpha(n)} \) where \( \alpha \) is the permutation of \{\( -8, -7, \cdots, 7, 8 \)\} induced by \( A \) and where \( \delta_{(−1)^p+q}\delta_{ij} = (−1)^{p+q}\delta_{ij} \).

\( D \) operating on (6) provides
\[ \delta_{66} = -\delta_{23} - \delta_{41}, \quad \delta_{66} = \delta_{22} + \delta_{44}, \quad \delta_{67} = \delta_{21} - \delta_{46}, \quad \delta_{68} = \delta_{26} + \delta_{43}, \]
and (7) gives
\[ \delta_{77} = \delta_{66} + \delta_{22}, \quad \delta_{78} = \delta_{24}. \]

From the vanishing of \([e_1, e_6]\) follows \( \delta_{12} = 0, \delta_{14} = -\delta_{65} \) and from \([e_2, e_5] = 0, \delta_{21} = 0, \delta_{23} = -\delta_{66} \). Again, another set of equations is obtained by applying \( A \).

Among the ten relations of the form \( \delta_{ii} + \delta_{jj} = \delta_{kk} \), eight are linearly independent so that \( \delta_{ii} = 0 \) for \( i = 1, 2, \cdots, 8 \). The relations
\[ \delta_{78} = \delta_{31} = \delta_{24} = \delta_{16} - \delta_{46} \]
and
\[ \delta_{57} = \delta_{45} - \delta_{31} = \delta_{16} + \delta_{24} \]
imply
\[ \delta_{67} = \delta_{16} = \delta_{46} \quad \text{and} \quad \delta_{24} = \delta_{78} = \delta_{31} = 0, \]
and there are also
The derivation algebra \( \mathfrak{D} \) is 12 dimensional and the algebra \( \mathfrak{Z} \) of inner derivations 6 dimensional. Every linear transformation sending \( \mathfrak{L} \) into the center of \( \mathfrak{L} \) and \( \mathfrak{L}^2 \) into 0 is a derivation. The ideal \( \mathfrak{D}_0 \) of these derivations is 8 dimensional and intersects \( \mathfrak{Z} \) in a 2 dimensional space. In a sense then \( \mathfrak{L} \) has as few outer derivations as possible. More precisely,

\[
\mathfrak{N} = \mathfrak{D}_0 + \mathfrak{Z}.
\]

The existence of \( \mathfrak{L} \) suggests the consideration of a subclass of nilpotent algebras which might prove more tractable than the entire class. To this end, for any algebra \( \mathfrak{N} \) with derivation algebra \( \mathfrak{D} \), let

\[
\mathfrak{N}^{[1]} = \mathfrak{N}\mathfrak{D} = \{ \sum x_i D_i \mid x_i \in \mathfrak{N}, D_i \in \mathfrak{D} \},
\]

and let

\[
\mathfrak{N}^{[k+1]} = \mathfrak{N}^{[k]}\mathfrak{D}.
\]

\( \mathfrak{N} \) could be called \textit{characteristically nilpotent} if for some \( k \), \( \mathfrak{N}^{[k]} = 0 \). The algebra \( \mathfrak{L} \) is characteristically nilpotent and for any such algebra \( \mathfrak{N} \),

(1) if \( \mathfrak{N} \) is an ideal of a solvable algebra \( \mathfrak{R} \) then either \( \mathfrak{N}^k \subset \mathfrak{N} \) for any \( k \) or \( \mathfrak{N} \) is nilpotent,
(2) if $M$ is an algebra with nil-radical $N$ then $M = N \oplus S$ for some semi-simple ideal $S$.

One might ask whether there is an intrinsic characterization of such algebras, and a general method for constructing them all.

The algebra $L$ has an additional property which may be shared by all characteristically nilpotent algebras: $L$ is not the derived algebra of any Lie algebra. To see this, observe that $L^{[1]} = L^2 = L^3$ so that if $L \subset M$ and $M^2 = L$, then $[LM] \subset L^{[1]} = L^2$. This implies $L^2 = [L[M|]] \subset [[LM]|M] \subset [L^2M] \subset [L^2L]$, and this contradicts the nilpotency of $L$.

Reference