1. Introduction. Almost left alternative algebras were defined by Albert in [1]. They are algebras $A$ over a field $F$ of characteristic not two which satisfy these postulates:

I. The elements of $A$ satisfy an identity of the form

\[ z(xy) = \alpha(zx)y + \beta(zy)x + \gamma(xz)y + \delta(yz)x + \epsilon y(zx) \]
\[ + \eta x(zy) + \sigma y(xs) + \tau x(yz) \]

(1)

for elements $\alpha, \beta, \gamma, \delta, \epsilon, \eta, \sigma, \tau$ in $F$ which are independent of $x, y, z$ in $A$.

II. The relation $xx^2 = x^2x$ holds for every $x$ of $A$.

III. There exists an algebra $B$ with a unity quantity $e$ such that $B$ satisfies (1) and is not a commutative algebra.

An algebra is called almost right alternative if I, II, and III hold with (1) replaced by an identity of the same form but with $z(xy)$ replaced by $(xy)z$. These two identities are the general shrinkability conditions of level one, as defined by Albert in [2]. An almost alternative algebra is one which is both almost left alternative and almost right alternative.

Reference is made in [1] to several results which are proved here. In addition to the above postulates, we assume the flexible law, that is, $(xy)x = x(yx)$ for every $x$ and $y$ in $A$. This makes Postulate II redundant. Albert confined his investigation in [1] to nonflexible algebras.
It is shown in this paper that flexible almost left alternative algebras are also almost right alternative, power-associative, Jordan-admissible, and that they are quasiequivalent to alternative algebras except when the parameter $\alpha + \beta = \frac{3}{4}$. Results for the latter case will be presented in another paper. The class of flexible almost left alternative algebras includes quasiasociative algebras. It is included in the class of noncommutative Jordan algebras as defined in [3].

2. Almost left alternative algebras. From Postulate III, if we replace $x$, $y$, $z$ in (1) by $e$, we obtain

$$\alpha + \beta + \gamma + \delta + \varepsilon + \eta + \sigma + \tau = 1.$$  

Next replace only $z$ by $e$. Then $(1 - \alpha - \gamma - \eta - \tau)xy = (\beta + \delta + \varepsilon + \sigma)yx$ and (2) implies that $(1 - \alpha - \gamma - \eta - \tau)(xy - yx) = 0$. Since $B$ in Postulate III is not commutative, $xy \neq yx$ for some $x$ and $y$. Hence

$$\alpha + \gamma + \eta + \tau = 1, \quad \beta + \delta + \varepsilon + \sigma = 0.$$  

Now replace $x$ by $e$. Then $(1 - \alpha - \beta - \gamma - \eta)(zy - yz) = 0$, and so

$$\alpha + \beta + \gamma + \eta = 1, \quad \delta + \varepsilon + \sigma + \tau = 0.$$  

Finally replace $y$ by $e$. Then $(1 - \alpha - \beta - \delta - \varepsilon)(xz - xz) = 0$, and so

$$\alpha + \beta + \delta + \varepsilon = 1, \quad \gamma + \eta + \sigma + \tau = 0.$$  

From (4) and (5),

$$\epsilon = 1 - \alpha - \beta - \delta, \quad \eta = 1 - \alpha - \beta - \gamma.$$  

From (3) and (6),

$$\sigma = \alpha - 1, \quad \tau = \beta.$$  

Hence there are only four essential parameters in (1).

The identity (1) may be expressed in terms of right and left multiplications of $A$ and is equivalent to

$$R_{xy} = \alpha R_x R_y + \beta R_y R_x + \gamma L_x R_y + \delta L_y R_x + \varepsilon R_z L_y + \eta R_y L_z$$  

$$+ \sigma L_z L_y + \tau L_y L_z.$$  

Interchanging $x$ and $y$, we obtain

$$R_{yz} = \alpha R_y R_z + \beta R_z R_y + \gamma L_y R_z + \delta L_z R_y + \epsilon R_y L_z + \eta R_z L_y$$  

$$+ \sigma L_y L_z + \tau L_z L_y.$$  

If we interchange $y$ and $z$ in (1), the resulting identity is equivalent to

$$L_x L_y = \alpha L_y L_x + \beta L_x L_y + \gamma L_{xz} + \delta R_y R_z + \epsilon R_{yz} + \eta L_y L_z$$  

$$+ \sigma R_{xz} + \tau R_y R_z.$$
Interchanging $x$ and $y$ in (10), we obtain

$$L_y L_z = \alpha L_{zy} + \beta L_z R_y + \gamma L_{yz} + \delta R_z R_y + \epsilon R_{zy} + \eta L_z L_y$$

(11)

$$+ \sigma R_{yz} + \tau R_z L_y.$$ 

If we interchange $x$ and $z$ in (1), the resulting identity is equivalent to

$$R_y L_z = \alpha L_x R_y + \beta L_{xy} + \gamma R_x R_y + \delta L_{zy} + \epsilon L_z L_y + \eta R_{xy}$$

(12)

$$+ \sigma R_z L_y + \tau R_{zy}.$$ 

Interchanging $x$ and $y$ in (12), we obtain

$$R_x L_y = \alpha L_y R_x + \beta L_{yx} + \gamma R_y R_x + \delta L_{xy} + \epsilon L_y L_x + \eta R_{yx}$$

(13)

$$+ \sigma R_x L_x + \tau R_{xy}.$$ 

Now add equations (8) through (13). The result together with (2) is equivalent to

$$(\alpha + \beta + \gamma + \delta)(R_{zy+yz} - L_{zy+yz} - R_z R_y - R_y R_z + L_z L_y$$

(14)

$$+ L_y L_z + R_y L_z + R_z L_y - L_z R_y - L_y R_z) = 0.$$ 

Then if $\alpha + \beta + \gamma + \delta \neq 0$, we obtain

$$R_{zy+yz} - L_{zy+yz} = (R_z + L_z)(R_y - L_y) + (R_y + L_y)(R_z - L_z).$$

When the characteristic of $F$ is prime to six, equation (15) implies that $xx^2 = x^2 x$ [2, p. 555] and so Postulate II is redundant.

By combining equations (8) through (13) in a different way, we obtain

$$R_{zy+yz} - L_{zy+yz} = (L_z L_y + L_y L_z - R_y L_z + R_z L_y)$$

(16)

$$= (\alpha - \beta - \gamma + \delta)(L_{zy-yz} + R_z R_y - R_y R_z - L_z R_y + L_y R_z)$$

$$+ (\epsilon - \eta - \sigma + \tau)(R_{zy-yz} - L_z L_y + L_y L_z - R_y L_z + R_z L_y).$$ 

Using (6) and (7), we see that (16) is equivalent to

$$(\alpha - \beta - \gamma + \delta)(R_{zy-yz} - L_{zy-yz} - L_z L_y + L_y L_z - R_z R_y$$

(17)

$$+ R_z R_y - R_y L_z + R_z L_y + L_z R_y - L_y R_z) = 0.$$ 

Then if $\alpha - \beta - \gamma + \delta \neq 0$, we obtain

$$R_{zy-yz} - L_{zy-yz} = (R_z - L_z)(R_y - L_y) - (R_y - L_y)(R_z - L_z).$$

This is a necessary and sufficient condition that the algebra $A$ be Lie-admissible [2, p. 575], that is, that the algebra $A^-$ in which the product is $[x, y] = xy - yx$ be a Lie algebra. We have proved

**Theorem 1.** If $A$ is an almost left alternative algebra such that $\alpha - \beta - \gamma + \delta \neq 0$, then $A$ is Lie-admissible.
3. Flexible algebras. In terms of right and left multiplications, the flexible law is the assumption that $R_xL_x = L_xR_x$ for every element $x$ of the algebra. Consequently, $R_{x+y}L_{x+y} = L_{x+y}R_{x+y}$, and so

$$R_xL_y + R_yL_x = L_xR_y + L_yR_x.$$  

This may also be written in the forms

$$R_xL_y - L_yR_x = L_xR_y - R_yL_x,$$

and

$$y(zx) - (yz)x = (xz)y - x(zy).$$

Interchanging $z$ and $y$, (21) is equivalent to

$$R_yz - R_zy = L_yx - L_xz.$$  

From this it follows that every flexible almost left alternative algebra is almost right alternative and hence almost alternative.

Using (6), (7), and (20), identity (8) for almost left alternative algebras can be written in the form

$$R_{xy} - R_xR_y = (x - 1)(R_xR_y + L_xL_y) + \beta(R_yR_z + L_yL_z)$$

$$+ (1 - \alpha - \beta)(R_xL_y + R_yL_z)$$

$$+ (\gamma - \delta)(L_xR_y - R_yL_z).$$

If $y = x$,

$$R_{xx} - R_xR_x = (\alpha + \beta - 1)(R_x - L_x)^2$$

so that $A$ is alternative if and only if $\alpha + \beta = 1$ or $(R_x - L_x)^2 = 0$.

**Theorem 2.** Let $A$ be a flexible almost left alternative algebra which is not alternative. Then $(\alpha - \beta + \gamma - \delta)(\alpha + \beta - 1) = \beta$.

**Proof.** Take $y = x$ in (10) and use (6), (7), and the flexible law to obtain

$$L_x^2 = (\alpha + \gamma) L_{xx} - (\beta + \delta) R_{xx} + \delta R_x^2 + (1 - \alpha - \beta - \gamma) L_x^2 + 2\beta R_xL_x.$$  

From (22) it follows that $L_{xx} = R_{xx} - R_x^2 + L_x^2$. Then $(\alpha - \beta + \gamma - \delta) \cdot (R_{xx} - R_x^2) = (R_x^2 + L_x^2 - 2R_xL_x)$. Combining with (24) we obtain 

$$[(\alpha - \beta + \gamma - \delta)(\alpha + \beta - 1) - \beta] \cdot (R_x - L_x)^2 = 0.$$  

Since $A$ is assumed to be not alternative, $(R_x - L_x)^2 \neq 0$ and the identity follows.

Because of flexibility, $R_x$ commutes with $L_x$. For this reason, any algebra which is flexible and shrinkable of level one satisfies the relation $R_xR_{xx} = R_{xx}R_x$, since $R_{xx}$ is expressible in terms of $R_x$ and $L_x$. This relation and that of flexibility are the postulates for noncommutative Jordan algebras, and as observed by Schafer in [3] such algebras with characteristic not two are Jordan-admissible and power-associative.
Theorem 3. Every flexible almost left alternative algebra $A$ with $\alpha + \beta \neq 3/4$ is quasiequivalent in a scalar extension $K$ of $F$ to an alternative algebra $B = A_K(\lambda)$.

Proof. Let $K = F(\xi)$ where $\xi^2 = 4\alpha + 4\beta - 3$ and let $\lambda = (\xi + 1)/2$. Since $\alpha + \beta \neq 3/4$, $\xi \neq 0$. Thus $\lambda$ is a quantity of $K$ different from $1/2$. Form $A_K(\lambda) = B$. Since $A$ is flexible, so is $A_K$. As shown in [2], $A_K(\lambda)$ is also flexible. Hence to show $B$ alternative, it is sufficient to show $R_x^2 = R_x^2$, where $R_x'$ is a right multiplication of $B$. Since $R_x' = \lambda R_x + (1 - \lambda)L_x$, $R_x^2 = \lambda R_x x + (1 - \lambda) L_x x = R_x x + (1 - \lambda)(L_x^2 - R_x^2)$ using (22); and $R_x^2 = \lambda^2 R_x^2 + 2\lambda(1 - \lambda) R_z x + (1 - \lambda)^2 L_x^2$.

Now $1 - \lambda = (1 - \xi)/2$ and $\alpha + \beta - 1 = (\xi^2 - 1)/4 = -\lambda(1 - \lambda)$. Thus (24) becomes $R_x^2 = R_x^2 - \lambda(1 - \lambda)(R_z - L_z)^2$, so that $R_x^2 = R_x^2 - \lambda(1 - \lambda) \cdot (R_z - L_z)^2 + (1 - \lambda)(L_x^2 - R_x^2) = \lambda^2 R_x^2 + 2\lambda(1 - \lambda) R_z L_z + (1 - \lambda)^2 L_x^2 = R_x^2$.

Corollary. A flexible almost left alternative algebra $A$ with $\alpha + \beta \neq 3/4$ over a field $F$ of characteristic prime to 6 is quasiassociative if and only if $A$ is Lie-admissible.

Proof. It is shown in [2] that an alternative algebra over a field of characteristic not 2 or 3 is associative if and only if it is Lie-admissible; and furthermore that $B$ is Lie-admissible if and only if $B(\mu)$ is Lie-admissible. Now $A_K = B(\mu)$ where $\mu = \lambda/(2\lambda - 1)$ and $A_K$ is Lie-admissible if and only if $A$ is Lie-admissible.

References