NOTE ON SOME GAP THEOREMS

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Many theorems in the theory of series concern gap series of the form

\[ \sum_{m=0}^{\infty} a_m \]

with \( a_m = 0 \) for \( m_k < m \leq M_k \) and \( m_k \uparrow \infty \),

where \( M_k \geq m_k (1 + \vartheta) \) for some \( \vartheta > 0 \),

with partial sums \( s_m \). These theorems infer the convergence of the sequence \( \{s_{m_k}\} \) for \( k \to \infty \) from assumptions concerning the summability of (1) or from properties of the associated complex function

\[ f(z) = \sum_{m=0}^{\infty} a_m z^m \quad (z = x + iy). \]

Two theorems of the latter type are the following Theorems A and B.\(^1\)

**Theorem A** \([1]\). Suppose that a series (1) is given and that (2) is regular in \( |z| < 1 \) and continuous in a circle \( |z - \alpha| \leq 1 - \alpha \) for some \( \alpha \) with \( 0 < \alpha < 1 \). Then \( s_{m_k} \to f(1) \) \((k \to \infty)\).\(^2\)

**Theorem B** \([4]\). Suppose that a series (1) is given and that (2) is regular in \( |z| < 1 \) and bounded in a sector \( |\arg z| < \epsilon, 0 <|z| < 1 \). Then \( \lim_{x \to 0} f(x) = s \) implies \( s_{m_k} \to s \) \((k \to \infty)\).

We are going to prove a theorem which contains both of these theorems:

**Theorem 1.** Suppose that a series (1) is given and that (2) is regular in \( |z| < 1 \) and bounded in a circle \( |z - \alpha| < 1 - \alpha \) for some \( \alpha \) with \( 0 < \alpha < 1 \). Then \( \lim_{z \to 1 - 0} f(x) = s \) implies \( s_{m_k} \to s \) \((k \to \infty)\).

By a short complex variable argument, we shall reduce the proof of Theorem 1 to the following gap theorem on summable series of type (1).

**Theorem 2.** Suppose that a series (1) is given and that

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1 In this connection, see also some gap theorems by Noble \([8; 9]\).

2 Evgrafov has also given an example to show that Theorem A may be false if \( |z - \alpha| \leq 1 - \alpha \) is replaced by a sector \( \pi/2 + \delta \leq \arg (z-1) \leq 3\pi/2 - \delta, 0 \leq |z-1| \leq \rho \) \((\delta > 0, \rho > 0)\).
(i) the function (2) is regular in \(|z| < \alpha\) and at \(z = \alpha\) for some \(\alpha\) with \(0 < \alpha < 1\);

(ii) at the point \(z = 1\), the Taylor series of \(f(z)\) about \(z = \alpha\) is \(C_1\)-summable to the value \(s\).

Then there exists a number \(\delta = \delta(\theta, \alpha) > 0\) such that under the additional hypothesis

(iii) \(s_m = O((1 + \delta)^m)\) \((m \to \infty)\)

one can conclude

\(s_{mk} \to s\) \((k \to \infty)\).

Remarks. (a) An equivalent form of (ii) is

(ii') the series (1) is \(C_1 T_\alpha\)-summable to the value \(s\), where \(T_\alpha\) denotes the \("circle method"\) of order \(\alpha\).\(^3\)

(b) Our proof of Theorem 2 goes beyond the theorem; it establishes the following extension:

The conclusion (3) in Theorem 2 remains valid if \(C_1\)-summability in (ii) or (ii') is replaced by \(C_k\)-summability; here \(k\) can be any number \(\geq 0\).

For \(\kappa = 0\) a similar theorem, with \(B\) (Borel) instead of \(T_\alpha\), has been proved by Zygmund (see [5, p. 206]); our theorem or its extension can, like Zygmund's theorem, be used for a proof of Ostrowski's theorem on overconvergence. Theorems similar to Theorem 2 and its extension could also be obtained by our method of proof for \(B\) (Borel) and \(E_p\) (Euler-Knopp) instead of \(T_\alpha\).

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Proof of Theorem 1. We show that all conditions of Theorem 2 are satisfied. Consider the development \(\sum_{m=0}^\infty a'_m(z-\alpha)^m\) of \(f(z)\) at \(z = \alpha\). Since \(f(z)\) is bounded for \(|z-\alpha| < 1-\alpha\), the Abel-summability of \(\sum_{m=0}^\infty a'_m(1-\alpha)^m\) to \(s\) implies the \(C_1\)-summability of the series to the same value (see for example [2, p. 327]). The condition (iii) of Theorem 2 is also satisfied for every \(\delta > 0\); hence Theorem 2 is applicable and it is sufficient to prove Theorem 2.

Proof of Theorem 2. We shall use our hypotheses in the form (i), (ii'), (iii). Let \(\{s'_n\}\) be the \(T_\alpha\)-transform of \(\{s_m\}\), i.e.

\[(4) \quad s'_n = (1 - \alpha)^n+1 \sum_{m=n}^\infty \binom{m}{n} \alpha^{m-n}s_m = \sum_{m\geq n} u_m(n)s_m,\]

with the notation of [5, p. 201] and \(k = 1-\alpha\), so that, for each fixed value \(n\), the maximum term \(u_m(n)\) is attained for \(m = [n/(1-\alpha)]\). Our idea is

\(^3\) To obtain the matrix for the \("circle method"\) of order \(\alpha\), put \(k = 1-\alpha\) in [5, p. 201].
(a) to construct for each interval \((m_k, M_k)\) in which \(\{s_m\}\) is constant \(= s_{mk}\), a corresponding interval \((n_k, N_k)\) in which \(\{s'_n\}\) is almost constant and almost \(= s_{mk}\) (see equation (12));

(b) to deduce from the \(C_1\)-summability of \(\{s'_n\}\) the convergence of \(\{s_{nk}\}\) and thus the convergence of \(\{s_{mk}\}\) for \(k \to \infty\).

In order to attack (a) for a given \(k = 1, 2, \ldots\), put \(\vartheta' = \vartheta/16\) and consider the intervals \(I_1^{(k)} = (m_k, [m_k(1+\vartheta')]), \ldots, I_{16}^{(k)} = ([m_k(1+15\vartheta'), [m_k(1+16\vartheta')])\).

For each \(k\), let \(n_k\) be the first index \(n\) for which \(\left[n/(1-\alpha)\right]\) is in \(I_1^{(k)}\); such an index exists for all sufficiently large \(k\). Similarly, let \(N_k\) be the last index \(n\) such that \(\left[n/(1-\alpha)\right]\) is in \(I_{16}^{(k)}\); again, such an index exists for large \(k\). Hence we have obtained our intervals \((n_k, N_k)\), and we note that by construction

\[
\left[\frac{N_k}{1-\alpha}\right] \geq [m_k(1 + 14\vartheta')]
\]

and

\[
[m_k(1 + \vartheta')] \leq \left[\frac{n_k}{1-\alpha}\right] \leq [m_k(1 + 2\vartheta')]
\]

so that

(5) \(N_k - n_k \geq \kappa n_k\) for every fixed positive \(\kappa < 12\vartheta'/(1 + 2\vartheta')\).

Consider now for any \(n\) in the interval \((n_k, N_k)\) the \(T_{\alpha}\)-transform \((4)\) and choose a positive number \(\sigma < \vartheta'/(1-\alpha)(1+\vartheta)\). For each such \(n\) the coefficients \(a_m\) in the interval \([n/(1-\alpha)] - \sigma n \leq m \leq [n/(1-\alpha)] + \sigma n\) vanish, for this implies \(m_k < m \leq M_k\). Therefore, putting \(m = [n/(1-\alpha)] + \vartheta\), we can estimate

(6) \(s'_n = \sum_{m \geq n} u_m(n)s_m = \sum_{|h| \leq \sigma n} u_m(n)s_m + \sum_{|h| > \sigma n} u_m(n)s_m = A_n + B_n\).

First, let us consider \(A_n\). We have

(7) \(A_n = s_{mk} \cdot \sum_{|h| \leq \sigma n} u_m(n) = s_{mk} \cdot \left(1 - \sum_{|h| > \sigma n} u_m(n)\right) = s_{mk} \cdot (1 + O(e^{-\gamma n}))\)

with some \(\gamma = \gamma(1-\alpha, \sigma) > 0\) (notation of Theorem 139 in [5]).

Next, we estimate \(B_n\) with the use of (iii). With a constant \(K\) we obtain

\[
|B_n| \leq K \sum_{|h| > \sigma n} u_m(n)(1 + \vartheta)^m
\]

\[
= K(1 - \alpha) \left(\frac{1 - \alpha}{\alpha}\right)^n \sum'\left(\frac{m}{n}\right)\{\alpha(1 + \vartheta)\}^m,
\]

* Incidentally, the assumption (ii) is not needed to prove (12).
where \( \sum' \) ranges over all \( m \) with \( m < \lfloor n/(1-\alpha) \rfloor - \sigma n \) and \( m > \lfloor n/(1-\alpha) \rfloor + \sigma n \). Put \( \alpha' = (1+\delta)\alpha \) and assume that \( \delta \) in (iii) is so small that

\[
(1 + \delta)\alpha < 1 \quad \text{and} \quad 0 < \frac{1}{1-\alpha} - \frac{1}{1-\alpha'} < \frac{\sigma}{2}.
\]

Then we have

\[
| B_n | \leq K(1-\alpha) \left( \frac{1-\alpha}{\alpha} \right)^n \sum'' \left( \begin{array}{c} m' \\ n \end{array} \right) \alpha'^m,
\]

where \( \sum'' \) ranges over all \( m \) with \( m < \lfloor n/(1-\alpha') \rfloor - (\sigma/2)n \) and \( m > \lfloor n/(1-\alpha') \rfloor + (\sigma/2)n \). This sum we can estimate:

\[
\sum'' \left( \begin{array}{c} m \\ n \end{array} \right) \alpha'^m = O(e^{-\gamma n}) \cdot \left( \frac{\alpha'}{1-\alpha'} \right)^n \quad \text{with} \quad \gamma = \gamma \left( 1 - \alpha', \frac{\sigma}{2} \right),
\]

so that

\[
| B_n | \leq K' \cdot \left( \frac{1-\alpha}{1-\alpha'} \cdot \frac{\alpha'}{\alpha} \right)^n \cdot e^{-\gamma n} \quad \text{with} \quad \gamma = \gamma \left( 1 - \alpha', \frac{\sigma}{2} \right).
\]

Now one can easily verify that \( \gamma(1-\alpha', \sigma/2) \) can be taken as a continuous function of \( \alpha' \) in \( 0 < \alpha' < 1 \), if \( \sigma > 0 \) is fixed. For \( \alpha' \to \alpha \) it tends therefore to \( \gamma(1-\alpha, \sigma/2) > 0 \), while the content of the braces in (9) tends to 1. Hence, if our \( \delta \) was in addition to (8) small enough (depending on \( \alpha \) and on \( \sigma \) and thus on \( \alpha \) and on \( \delta \)), we have

\[
| B_n | \leq K'' \cdot e^{-\lambda n} \quad \text{for some fixed constants} \quad K'' > 0 \quad \text{and} \quad \lambda_1 > 0;
\]

this holds for every \( n \) in the intervals \( (n_k, N_k) \).

To bring (7) into a more suitable form, we notice that, by (iii),

\[
\sum_{m_k} = O((1+\delta)^{m_k}),
\]

and hence

\[
\sum_{m_k} \cdot O(e^{-\gamma n}) = O((1 + \delta)^{m_k} e^{-\gamma n}) = O((1 + \delta)^{n/(1-\alpha)} e^{-\gamma n})
\]

\[ = O(e^{-\lambda n}), \]

if \( \delta \) in (iii) was given so that \( \log (1+\delta) < (1-\alpha) \cdot \gamma(1-\alpha, \sigma) \). Combining (6), (7), (10), (11), we obtain

\[
s'_n = s_{m_k} + O(e^{-\lambda n}) \quad \text{for a constant} \quad \lambda > 0 \quad \text{and for all} \quad n \in \text{the intervals} \quad (n_k, N_k).
\]

Now we come to the easier part (b) of our program. Denoting the \( C_1 \)-means of \( \{s'_n\} \) by \( \sigma'_n \), we simply write
\[
\sigma'_{N_k} = \frac{s'_0 + \cdots + s'_{N_k}}{N_k + 1} = \frac{s'_0 + \cdots + s'_{n_k}}{N_k + 1} + \frac{s'_{n_k+1} + \cdots + s'_{N_k}}{N_k + 1}
\]

(13)

\[
= \frac{n_k + 1}{N_k + 1} \sigma'_{n_k} + \frac{N_k - n_k}{N_k + 1} s_{m_k} + O(e^{-\lambda n_k}).
\]

Assuming without loss of generality that \(\sigma'_n \to 0 \ (n \to \infty)\) and remembering (5), we obtain immediately \(s_{m_k} \to 0 \ (k \to \infty)\), as was to be proved.

In order to prove the extension, replace the simple device (13) by the gap theorem for \(C_\varepsilon\)-summable series (see [7, p. 469]; for a simpler proof see [6] or [3]); this gap theorem is applicable to the series \(\sum (s'_n - s'_{n-1})\) because of (12), and it yields the convergence of \(\{s'_{n_k}\}\) and hence the convergence of \(\{s_{m_k}\}\). Thus our extension is also proved.

**References**


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