REFERENCES

1. W. Ambrose, *Structure theorem for a special class of Banach algebras*, Trans. Amer. Math. Soc. vol. 57 (1945) pp. 364–386.
2. N. Jacobson, *The radical and semi-simplicity for arbitrary rings*, Amer. J. Math. vol. 67 (1945) pp. 300–320.
3. I. Kaplansky, *Dual rings*, Ann. of Math. vol. 49 (1948) pp. 689–701.
4. L. H. Loomis, *An introduction to harmonic analysis*, New York, Van Nostrand, 1953, pp. 100–106.
5. P. P. Saworotnow, *On a generalization of the notion of H^*-algebra*, Proc. Amer. Math. Soc. vol. 8 (1957) pp. 49–55.
6. M. F. Smiley, *Right H^*-algebra*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 1–4

CATHOLIC UNIVERSITY OF AMERICA

A BOUNDARY LAYER PROBLEM FOR AN ELLIPTIC EQUATION IN THE NEIGHBORHOOD OF A SINGULAR POINT[1]

VICTOR J. MIZEL

We consider the first boundary value problem for

$$Lu = \epsilon\Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y)$$

on a region R under the following hypotheses

I. R is an open simply- or multiply-connected region in the (x, y) plane whose boundary S consists of a finite number of simple closed curves, and $R+S$ is contained in an open connected region R_0 throughout which $A(x, y)$, $B(x, y)$, $C(x, y)$, and $D(x, y)$ are of class C^6.

II. Along each closed curve of S the functions giving x, y, and the boundary value $\bar{u}$ in terms of arclength are of class C^6.

III. $C(x, y) < 0$ on R_0.

IV. The system (for characteristics of the abridged ($\epsilon = 0$) equation)

$$(1) \qquad \frac{dx}{dt} = -A(x, y), \qquad \frac{dy}{dt} = -B(x, y)$$

has as its singularities on $R+S$ a finite number of stable attractors $P_1, \cdots, P_n$.

Received by the editors February 14, 1956.

[1] The author wishes at this point to express his gratitude to Professor N. Levinson who originally suggested the problem to him and who gave him encouragement throughout.

We shall prove that if $u(x, y, \epsilon)$ is the solution to our boundary value problem (existence for small $\epsilon > 0$ follows from results of Lichtenstein [4]), and if $U(x, y)$ is that solution to the abridged equation

$$(2) \qquad L^0 U = A(x, y)U_x + B(x, y)U_y + C(x, y)U = D(x, y)$$

which solves the initial value problem for $R+S-P_1- \cdots -P_n$: $U=\bar{u}$ on those portions of S where the solutions of (1) cross *into* R, then throughout $R-P_1- \cdots -P_n$, $v(x, y, \epsilon) \equiv U(x, y) - u(x, y, \epsilon)$ approaches zero as $\epsilon \to +0$ except possibly, as will be seen from the proof, at characteristics of (1) which are somewhere tangent to S.

Now Levinson [3] has proved this in the case where (1) has no singularities on R. Indeed his results show in any case that for a certain set of subregions of $R+S$, the "regular quadrilaterals," the above stated conclusion is correct. More precisely, these "regular quadrilaterals" are defined by:

Let S_1 and S_2 be segments of curves of S having the property that they are nowhere tangent to a characteristic of (2) and being so related that those characteristics of (2) emanating from S_1 pass out of R on S_2 and conversely. Here S_1 signifies that one of the pair of segments across which these characteristics cross into R (referring to (1)). That closed simply-connected subregion of $R+S$ bounded by S_1, S_2, and the two characteristics of (2) joining their endpoints is a "regular quadrilateral." [Thus in our problem R cannot be decomposed as a union of regular quadrilaterals.]

Levinson's result then reads (in our notation):

THEOREM. *In a regular quadrilateral we may write*

$$v(x, y, \epsilon) = z(x, y, \epsilon) - w(x, y, \epsilon)$$

where

$$w = O(\epsilon^{1/2}) \qquad \textit{as } \epsilon \to +0$$

uniformly in the quadrilateral and $w=0$ *on* S_1 *and* S_2, *and where* $z(x, y, \epsilon)$ *has near and on* S_2 *the form* $e^{-g(x,y)/\epsilon} h(x, y)$. *Here* $g=0$ *on* S_2 *and* $g>0$ *off of* S_2 *and both* g *and* h *are of class* C^2; *moreover, at points of the quadrilateral where the above representation is not valid*

$$z = O(e^{-\delta/\epsilon}) \qquad \textit{uniformly as } \epsilon \to +0$$

for a fixed positive δ.

Therefore it will suffice for us to prove that the stated result holds for a second set of subregions of $R+S$, the "regular triangles," these being defined as follows:

Let S_0 be a closed segment of a curve of S having the property that it is nowhere tangent to a characteristic of (2) and such that those characteristics emanating from S_0 enter into and remain in R (referring to (1)), where they approach one of the P_i. That simply-connected subregion of $R+S$ traced out by the characteristics emanating from S_0 is a "regular triangle."

Moreover to show that, on any regular triangle T, $v(x, y, \epsilon)$ approaches zero as $\epsilon \to +0$ it will clearly be sufficient to show that $v(x, y, \epsilon)$ approaches zero as $\epsilon \to +0$ on all subregions G of the following type:

G is a simply-connected subregion of T which is bounded by S_0, by an orthogonal trajectory to those characteristics of (2) lying in T which intersects the two characteristics making up the "sides" of T but does not intersect S_0, and by the requisite portions of the "sides" of T.

We note, too, that the nontangency condition and the stability of the attractor allow us to consider two other triangles T_1, T_2 such that $T_2 \supset T_1 \supset T$ and corresponding subregions G_1, G_2 defined analogously to G (the orthogonal trajectory boundary for G_1 is taken to be a portion of that for G_2 and it lies "nearer" to the attractor than does that for G, so that $G_2 \supset G_1 \supset G$). We shall find it convenient to introduce characteristic coordinates on G_2. Levinson [3] has shown that there are C^6 functions $\sigma(x, y)$, $\tau(x, y)$ satisfying $A\sigma_x + B\sigma_y = 0$, $B\tau_x - A\tau_y = 0$ on a region such as G_2, such that: $\partial(\sigma, \tau)/\partial(x, y) \neq 0$; along characteristics of (2), $\sigma = \text{constant}$; and the curvilinear coordinates (σ, τ) are orthogonal. In addition $A\tau_x + B\tau_y < 0$ if τ is taken as increasing toward the singularity, as we shall do. We denote the values of σ on the characteristic boundaries of G_2 by $\bar{\sigma}_1$ and $\bar{\sigma}_2$ while those values on the characteristic boundaries of G_1 are denoted by σ_1 and σ_2 (ordering so that $\bar{\sigma}_2 > \sigma_2 > \sigma_1 > \bar{\sigma}_1$).

The proof proceeds in the following manner: Following a technique used by Kamenomostskaya [2] and Aronson [1], we exhibit functions W_1 and W_2 defined on G_1 which consist of "boundary layer" terms alone, except for terms which are $O(\epsilon^{1/2})$, and which satisfy

$$LW_1 = Lv \qquad LW_2 = -Lv = L(-v),$$

where we recall

$$v = U - u.$$

These functions are so chosen that $W_1, W_2 > |v|$ on the boundary of G_1 for sufficiently small $\epsilon > 0$. Use of the maximum principle then implies that for such ϵ, $W_1 > v$ and $W_2 > -v$ throughout G_1. Finally, inspection of W_1 and W_2 shows that they are uniformly $O(\epsilon^{1/2})$ on G, and this yields the desired result.

Now in the preceding outline of the proof, "boundary layer" terms denote functions $H(x, y, \epsilon)$ of the following type:

1. In a neighborhood of a portion of the boundary of the region, $H(x, y, \epsilon)$ is of the form $e^{-g(x,y)/\epsilon^m}h(x, y)$, where m is a positive constant and where g, h are of class C^2, g being positive except on this portion of the boundary. Moreover H is of class C^2 throughout the entire region and H is uniformly $o(1)$ as $\epsilon \to +0$, except in the boundary neighborhood.

2. $LH = o(1)$ as $\epsilon \to +0$, uniformly in the entire region.

From direct substitution it is readily seen that only for $m = 1/2$ and $m = 1$ does the second condition give rise to as few as two equations which the functions g and h must satisfy. For these values the equations to be satisfied are:

$$m = 1: \quad g_x^2 + g_y^2 - Ag_x - Bg_y = 0,$$

$$(A - 2g_x)hx + (B - 2g_y)hy + (C - \Delta g)h = 0,$$

$$(3) \quad m = 1/2: Ag_x + Bg_y = 0, \quad Ah_x + Bh_y + (C + g_x^2 + g_y^2)h = 0,$$

or, equivalently,

$$g_\tau = 0, (A\tau_x + B\tau_y)h_\tau + [C + g_\sigma^2(\sigma_x^2 + \sigma_y^2)]h = 0.$$

The case $m = 1$ was considered by Levinson in his proof of the theorem stated previously. From his work it follows that a boundary layer term having the indicated exponential form $(m = 1)$ near and on the orthogonal trajectory boundary of G_2 can be constructed, where the value of $h(x, y)$ can be specified on this boundary in any C^2 manner so long as it vanishes near the end points of this boundary. Indeed it is further true that

$$LH = O(\epsilon)$$

and that except for the boundary neighborhood involved H is uniformly $O(e^{-\delta/\epsilon})$ where δ is a fixed positive constant. For the case $m = 1/2$, on the other hand, it follows from (3) that g must simply be a function which is constant on each characteristic of (2), so that to obtain a "boundary layer" term by this scheme we require a characteristic boundary (that is, we can only require $g = 0$ on a boundary which is characteristic if we are to retain a bona fide boundary layer form). Moreover since h satisfies a linear equation (cf. (3)) whose characteristics coincide with those of (2) we may readily specify this function throughout G_1 as a solution to an initial value problem. It follows that in this case we can obtain a boundary layer term having the indicated exponential form throughout the region G_1.

We now consider a pair of functions W_1 and W_2 defined on G_1 and having the form:

$$W_i(\sigma, \tau, \epsilon) = H^{(0)}(\sigma, \tau, \epsilon) + h^{(1)}(\sigma, \tau)\left[e^{-k(\sigma-\sigma_1)/\epsilon^{1/2}} + e^{-k(\sigma_2-\sigma)/\epsilon^{1/2}}\right] + \epsilon^{1/2}Z_i, \qquad i = 1, 2. \tag{4}$$

In the above expression $H^{(0)}(\sigma, \tau, \epsilon)$ is chosen according to Levinson's method for the case $m=1$ to be a boundary layer term for the region G_2 corresponding to the following values on the orthogonal trajectory boundary (a curve $\tau=$constant) of G_2

$$H^{(0)} = \begin{cases} M+1, & \sigma_1 < \sigma < \sigma_2, \\ 0, \bar{\sigma}_1 < \sigma < \dfrac{\bar{\sigma}_1 + \sigma_1}{2}, \ \bar{\sigma}_2 > \sigma > \dfrac{\bar{\sigma}_2 + \sigma_2}{2}, & \\ \text{a positive } C^2 \text{ “tieup” function} & \dfrac{\bar{\sigma}_1 + \sigma_1}{2} \leqq \sigma \leqq \sigma_1, \\ & \dfrac{\bar{\sigma}_2 + \sigma_2}{2} \geqq \sigma \geqq \sigma_2 \end{cases} \tag{5}$$

[M is a uniform bound for $v(x, y, \epsilon)$ on G_2—use of the maximum principle extended to the inhomogeneous case [3] shows there is a uniform bound for $u(x, y, \epsilon)$]. As for $h^{(1)}(\sigma, \tau)$, we choose it to be that solution of (cf. (3))

$$(A\tau_x + B\tau_y)h_\tau + [C + k^2(\sigma_x^2 + \sigma_y^2)]h = 0 \tag{6}$$

which on the segment of S bounding G_1 takes on the value $M+1$, while k is a positive constant required to be sufficiently large so that the coefficient of h in (6) is positive throughout G_1. Thus $h^{(1)}(\sigma, \tau)$ is defined and $h^{(1)} \geqq M+1$ throughout G_1. Finally, we choose the Z_i so that

$$LW_1 = Lv = \epsilon\Delta U, \qquad LW_2 = L(-v) = -\epsilon\Delta U,$$

and Z_1 and Z_2 vanish on the entire boundary of G_1. Thus the Z_i must solve the homogeneous boundary value problem on G_1 for

$$\begin{aligned} \epsilon^{1/2}LZ_i = {} & \epsilon^{1/2}\left\{\left[2kh_\sigma^{(1)}(\sigma_x^2 + \sigma_y^2) + kh^{(1)}\Delta\sigma\right] \cdot \left(e^{-k(\sigma-\sigma_1)/\epsilon^{1/2}} - e^{-k(\sigma_2-\sigma)/\epsilon^{1/2}}\right)\right\} \\ & + \epsilon\left\{\left[h_{\sigma\sigma}^{(1)}(\sigma_x^2 + \sigma_y^2) + h_{\tau\tau}^{(1)}(\tau_x^2 + \tau_y^2) + h_\sigma^{(1)}\Delta\sigma + h_\tau^{(1)}\Delta\tau\right]\left(e^{-k(\sigma-\sigma_1)/\epsilon^{1/2}} - e^{-k(\sigma_2-\sigma)/\epsilon^{1/2}}\right)\right\} \\ & - LH^{(0)} + \epsilon(-1)^{i+1}\Delta U, \qquad i = 1, 2. \end{aligned} \tag{7}$$

(It follows from results of Lichtenstein that C^2 solutions to these boundary value problems exist [4].) It therefore follows from the inhomogeneous case maximum principle [3] that $Z_i = O(1)$ as $\epsilon \to +0$, uniformly on G_1 [recalling that $LH^{(0)} = O(\epsilon)$].

Now examination of the functions W_1 and W_2 indicates that in G $W_i = O(\epsilon^{1/2})$ uniformly as $\epsilon \to +0$. Moreover $W_1 - v$ and $W_2 + v$ satisfy the homogeneous equation $Lu = 0$, while investigation of the boundary values shows that, for sufficiently small ϵ, $W_1 - v$ and $W_2 + v$ are positive throughout the boundary of G_1. Thus the maximum principle shows that for these ϵ, $W_1 - v \geqq 0$ and $W_2 + v \geqq 0$ throughout G_1. In particular,

$$|v| \leqq W_1 + W_2 = O(\epsilon^{1/2}) \qquad \text{throughout } G,$$

which completes the proof.

Bibliography

1. D. Aronson, *Linear parabolic equations containing small parameters*, to appear in Journal of Rational Mechanics and Analysis.

2. S. L. Kamenomostskaya, *On equations of elliptic and parabolic type with a small parameter in the highest derivative*, Mat. Sbornik N.S. vol. 31(73) (1952) pp. 703–708. (Russian).

3. N. Levinson, *The first boundary value problem for* $\epsilon\Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y)$ *for small* ϵ, Ann. of Math. vol. 51 (1950) pp. 428–445.

4. L. Lichtenstein, *Randwertaufgaben der Theorie der linearen partiellen Differentialgleichungen zweiter Ordnung von elliptischen Typus*, J. Reine Angew. Math. vol. 142 (1913) pp. 1–40.

Massachusetts Institute of Technology