ON $H$-SPACES WITH TWO NONTRIVIAL HOMOTOPY GROUPS$^1$

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Introduction. Although several topologists (e.g. H. Hopf and A. Borel) have found necessary algebraic conditions for a space to admit an $H$-space structure, very little has been done towards obtaining sufficient conditions. The author believes that the present paper contains essentially the first result in the latter direction.

Let $Y$ be a topological space with $y_0 \in Y$, $Y \cup Y = Y \times y_0 \cup y_0 \times Y \subset Y \times Y$. If $\phi: Y \cup Y \to Y$ is the map given by $\phi(y, y_0) = (y_0, y) = y$, then the problem of finding an $H$-space structure on $Y$ may be expressed as the problem of extending $\phi$ to a map $\phi': Y \times Y \to Y$. It is found that if $Y$ is a 1-connected, locally finite CW-complex [3], the obstructions to extending $\phi$ may be expressed in terms of Postnikov invariants [4] and partial extensions of $\phi$. If $Y$ has only two nonzero homotopy groups then there is at most one nontrivial obstruction. This will be zero if and only if the Eilenberg-MacLane $k$-invariant of $Y$ is primitive.

The relation between the existence of an $H$-structure and the vanishing of the J. H. C. Whitehead bracket products is investigated. This leads to a description of the lowest-dimensional bracket products on spaces whose first two nontrivial homotopy groups are in dimensions $n$ and $2n-1$ ($n \geq 1$).

1. This section presents much of the notation to be used, some discussion preliminary to the main result of the paper and an example of a space which is not an $H$-space, and yet has trivial bracket products.

If $f_i: X_i \to Y_i$ ($i = 1, 2$) are maps then the map $f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$ is defined by $f_1 \times f_2(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for $x_i \in X_i$. If $g_1: Y_1 \to Z_1$, then $g_1 \circ f_1: X_1 \to Z_1$ is the map given by $g_1 \circ f_1(x) = g_1(f_1(x))$ for $x \in X$, $I = [0, 1]$ is the closed unit interval; $I^m = I \times \cdots \times I$ ($m$-factors) is the unit $m$-cube; $\partial I^m$ is the usual boundary set of $I^m$. The discussion is restricted to locally finite CW-complexes, for if $Y$ is a locally finite CW-complex then $Y \times Y$ is a CW-complex whose (closed) $m$-cells may be taken to be of the form $E^m = E^p \times E^q$ ($p+q = m$; $E^p$, $E^q$ are, respectively, $p$-, and $q$-cells of $Y$) [6]. The characteristic map of $E^m$ is $e^m = e^p \times e^q$; $I^m \to E^m$ where $e^p$, $e^q$ are character-

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istic maps of \( E^p, E^q \) and \( I^m \) is identified with \( I^p \times I^q \). Given cochains \( c^p \in C^p(Y; G_1), c^q \in C^q(Y; G_2) \) and a pairing \( \xi: G_1 \otimes G_2 \rightarrow G \) we form a cochain \( c^p \times c^q \in C^{p+q}(Y \times Y; G) \) whose value on a cell \( E^p \times E^q \) is \( c^p \times c^q(E^p \times E^q) = \xi(c^p(E^p) \otimes c^q(E^q)) \) and which is zero elsewhere. If \( c^p, c^q \) are cocycles then \( c^p \times c^q \) is, and the corresponding cohomology classes are \([c^p], [c^q], [c^p] \times [c^q] = [c^p \times c^q] \). If \( X \) is a CW-complex, its \( m \)-skeleton will be designated by \( X^m \). The map \( \phi: \overline{Y} \Rightarrow Y \), presented in the introduction, will be called the folding map of \( Y \). If \( f: A \rightarrow B \) is a map, its homotopy class will be designated by \([f] \).

If the folding map has been extended over \( Y \cup (Y \times Y)^m \) then the obstruction to extending over the \((m + 1)\)-skeleton is in the cohomology group \( H^{m+1}(Y \times Y, Y \cup Y; \pi_m(Y)) \).

In case \( Y \) is \((n-1)\)-connected, \((Y \times Y, Y \cup Y)\) will be \((2n-1)\)-connected.

**Proposition 1.** The obstruction in dimension \( 2n \) to extending the folding map is

\[
d^n \times d^n \in H^{2n}(Y \times Y, Y \cup Y; \pi_{2n-1}(Y))
\]

where \( d^n \in H^n(Y, y; \pi_n(Y)) \) is the characteristic class for \( Y \) and the pairing of \( \pi_n(Y) \otimes \pi_n(Y) \) into \( \pi_{2n-1}(Y) \) is the J. H. C. Whitehead bracket product.

**Proof.** We may replace \( Y \) by a space whose \((n-1)\)-skeleton is a point (see §3). Then the \((2n-1)\)-skeleton of \( Y \times Y \) is contained in \( Y \cup Y \) so that the obstruction cocycle to extending \( \phi \) is given by:

\[
c^{2n}(E^n_1 \times E^n_2) = \{ \phi \circ (e^n_1 \times e^n_2) | \overline{I}^{2n} \}
\]

where \( E^n_1, E^n_2 \) are cells of \( Y \) and \( e^n_1, e^n_2 \) their characteristic maps. Note that

\[
\{ \phi \circ (e^n_1 \times e^n_2) | \overline{I}^{2n} \} = \{ \{ e^n_1 \}, \{ e^n_2 \} \}
\]

with \( \{ e^n_1 \}, \{ e^n_2 \} \) regarded as elements of \( \pi_n(Y) \) [1]. But \( d^n \) is the class of the cocycle \( c^n \) given by

\[
c^n(E^n) = \{ e^n \}.
\]

Thus \( c^{2n} = c^n \circ c^n \).

**Corollary 2.** \( c^{2n} = 0 \) if and only if \([\alpha, \beta] = 0 \) for \( \alpha, \beta \in \pi_n(Y) \).

**Proof.** Suppose \( c^{2n} = 0 \). Since \( H_n(Y) \approx \pi_n(Y) \) the characteristic maps of \( n \)-cells generate \( \pi_n(Y) \). Thus \([\alpha, \beta] = 0 \) when \( \alpha, \beta \) are in this set of generators; hence \([\alpha', \beta'] = 0 \) for all \( \alpha', \beta' \in \pi_n(Y) \). The converse is trivial.
On the other hand, if $Y$ is an $H$-space then all bracket products vanish. This raises the question: if $Y$ is a $CW$-complex and $[\pi_p(Y), \pi_q(Y)] = 0$ for $p, q > 0$, is $Y$ an $H$-space? The answer is negative, and we present a counter-example:

We construct a $CW$-complex, $K$, by specifying its $m$-skeleta, $K^m$. Let $n > 2$ be an integer and $p$ an odd prime. The $2n$-skeleton of $K$ is taken to be the $2n$-skeleton of a $CW$-complex which is an $Eilenberg$-$Mac$-$Lane$ space of type $(Z_p, n)$, where $Z_p$ is the integers mod $p$. There are no cells in dimension $2n + 1$ ($K^{2n+1} = K^{2n}$) and in higher dimensions, cells are appended so that $\pi_i(K) = 0$ for $i > 2n$ [7].

This creates a space whose homotopy groups are trivial except in dimensions $n$ and $2n$. Thus all bracket products vanish. Also note that $H^n(K; Z_p) \approx H^{n+1}(K; Z_p) \approx Z_p$ since the cohomology groups in these dimensions are those of a space of type $(Z_p, n)$. The $(2n+1)$-cohomology group is zero, since there are no $(2n+1)$-cells.

Now, $K$ is not an $H$-space, for if it were, its cohomology ring would be a Hopf algebra and the cup product of an element of $H^n(K; Z_p)$ with an element of $H^{n+1}(K; Z_p)$ would be nonzero, contradicting $H^{n+1}(K; Z_p) = 0$ [2].

2. The main result. Let $Y$ be a 1-connected $CW$-complex, and suppose that the folding map has been extended to $\phi: Y \vee Y \cup (Y \times Y)^m \to Y$. Let $X$ be a $CW$-complex consisting of $Y$ united with $i$-cells, $E'$, $(i > m)$ such that $\pi_i(X) = 0$ for $i \geq m$. Note that below dimension $m$, the inclusion map induces isomorphisms of the homotopy groups of $X$ and $Y$.

**Proposition 3.** $X$ is an $H$-space.

**Proof.** The $m$-skeleta of $X$ and $Y$ are the same, so that $(X \times X)^m = (Y \times Y)^m$. Thus $\phi$ provides an extension $\psi': X \vee X \cup (X \times X)^m \to X$. But $H^{i+1}(X \times X, X \vee X; \pi_i(X)) = 0$ for $i > m$, whence all obstructions to extending $\psi'$ vanish. One such extension, let us call it $\psi: X \times X \to X$, is chosen for the structure map of $X$.

The condition $\pi_i(X) = 0$ for $i \geq m$ also implies that any two extensions of $\psi'$ will be homotopic.

In the diagram below, $i^*_1, \cdots, i^*_6$ are homomorphisms induced by the appropriate inclusion maps: $\psi^*, \psi^*_i$ are induced by $\psi$. The coefficient group for each of these cohomology groups is $\pi_m(Y)$. This symbol has been omitted to save space. It is well known that $i^*_i$ sends $H^{m+1}(X \times X, X \vee X; \pi_m(Y))$ isomorphically onto a direct summand of $H^{m+1}(X \times X; \pi_m(Y))$. This direct sum decomposition induces the homomorphism $\rho$. The composition $\rho \circ i^*_i$ is the identity automorphism of $H^{m+1}(X \times X, X \vee X; \pi_m(Y))$, and the kernel of $\rho$ is essen-
tially \( H^{m+1}(X \vee X; \pi_m(Y)) \). The diagram is commutative.

\[
\begin{array}{ccc}
H^{m+1}(X \times X) & \xrightarrow{i_*} & H^{m+1}(X \vee X, X \vee X) \\
\downarrow \rho & & \downarrow i_* \\
\downarrow \psi_* & & \downarrow i_* \\
H^{m+1}(X) & \xleftarrow{i_*} & H^{m+1}(X, Y)
\end{array}
\]

Let \( k' \in H^{m+1}(X, Y; \pi_m(Y)) \) be the first obstruction to retracting \( X \) onto \( Y \), and \( k = i_2^*k' \in H^{m+1}(X; \pi_m(Y)) \). The maps \( \rho_1, \rho_2 : X \times X \to X \) are the projections \( \rho_i(x_1, x_2) = x_i \), for \( x_i \in X \), \( i = 1, 2 \).

**Proposition 4.** Let \( \gamma \in H^{m+1}(Y \times Y, Y \vee Y; \pi_m(Y)) \) be the class of the obstruction to extending \( \phi \) to \( (Y \times Y)^{m+1} \cup Y \vee Y \). Then

\[
i_2^* \rho(\psi_* - \rho_1^* - \rho_2^*)k = \gamma.
\]

**Proof.** The cohomology class \( \psi_*k' \) is the first obstruction to extending \( (\psi_* Y \vee Y \cup (X \times X)^m) = (\phi \mid Y \vee Y \cup (X \times X)^m) \) to \( X \times X \) and hence \( i_4^* \psi_* k' = \gamma \). On the other hand, \( i_3^* \psi_* k' = \psi_* i_5^* k' = \psi_* k \), and so \( i_2^* \rho \psi_* k = i_2^* \rho i_3^* \psi_* k' = i_4^* \psi_* k' = \gamma \). Finally, \( \rho \circ (\rho_1^* + \rho_2^*) \) is the trivial homomorphism, whence \( i_2^* \rho (\psi_* - \rho_1^* - \rho_2^*) k = i_2^* \rho \psi_* k = \gamma \).

If \( W \) is an \( H \)-space with \( \psi, \rho_1, \rho_2 : W \times W \to W \) the structure map and the two projections respectively, then a cohomology class, \( u \), is called **primitive** whenever \( (\psi_* - \rho_1^* - \rho_2^*) u = 0 \).

**Theorem 5.** The obstruction class, \( \gamma \), vanishes if and only if \( k \) is primitive.

**Proof.** We already have \( i_2^* \rho (\psi_* - \rho_1^* - \rho_2^*) k = \gamma \) so that if \( k \) is primitive, \( \gamma = 0 \). To obtain the converse, we first note that \( i_2^* \) is an isomorphism since the inclusion map of \( Y \) in \( X \) induces isomorphisms \( H^i(X) \approx H^i(Y) \) for \( i \leq m \). But the image of \( (\psi_* - \rho_1^* - \rho_2^*) k \) in \( H^{m+1} \cdot (X \vee X; \pi_m(Y)) \) is zero, since \( \psi_* (X \vee X)^* = (\rho_1^* X \vee X)^* + (\rho_2^* X \vee X)^* \). Thus \( (\psi_* - \rho_1^* - \rho_2^*) k \in i_2^* H^{m+1}(X \times X, X \vee X; \pi_m(Y)) \) from which it follows that \( \rho (\psi_* - \rho_1^* - \rho_2^*) k = 0 \) implies \( (\psi_* - \rho_1^* - \rho_2^*) k = 0 \). This completes the proof.

Note that \( k \) is essentially the \((m+1)- Postnikov\) invariant of \( Y \). The theorem fails to provide a decisive victory over the problem of characterizing \( H \)-spaces which are CW-complexes, inasmuch as it depends upon choosing a particular extension, \( \psi \), in each dimension.

However, for sufficiently simple spaces the problem can be solved:

**Theorem 6.** Suppose \( Y \) is a CW-complex which has only two non-trivial homotopy groups, \( \pi_n(Y) \) and \( \pi_m(Y) \), with \( 1 < n < m \). Then \( Y \) is
an $H$-space if and only if the Eilenberg-MacLane $k$-invariant of $Y$ is primitive.

**Proof.** The only nontrivial obstruction to extending the folding map is in $H^{m+1}(X \times Y, Y \vee Y; \pi_m(Y))$. Thus the space $X$ is of type $(\pi_n(Y), n)$ and $k$ may be identified with the $k$-invariant, $k_n^{m+1}(Y)$. The result now follows from Theorem 4.

It may now be seen that the example of §1 was obtained by constructing a space with nonprimitive $k$-invariant.

In §3 it is shown that given abelian groups $\pi_n$, $\pi_m$ and an element $k \in H^{n+1}(\pi_n, n; \pi_m)$, there is a space $Y$ (which may be taken to be a CW-complex) such that $\pi_i(Y) = 0$ for $i < n$, $m$, $\pi_n(Y) = \pi_n$, $\pi_m(Y) = \pi_m$ and $k_{n+1}(Y) = k$. Any two such CW-complexes are of the same homotopy type. This observation and Theorem 5 give a classification of CW-complexes which admit $H$-structures and have only two nonvanishing homotopy groups.

Theorem 5 and Proposition 1 (§1) may be combined to yield a result about the Whitehead bracket product. Suppose $\pi_i(Y) = 0$ for $0 \leq i < n$ and $n < i < 2n - 1$. Let $\pi_n = \pi_n(Y)$ and $\pi_{2n-1} = \pi_{2n-1}(Y)$; $\pi_n \otimes \pi_n$, $\pi_n \oplus \pi_n$ designate respectively the tensor product and the direct sum of $\pi_n$ with itself. Three homomorphisms, $\tilde{\psi}$, $\tilde{p}_1$, $\tilde{p}_2 : \pi_n \otimes \pi_n \rightarrow \pi_n$ are defined by $\tilde{\psi}(\alpha, \beta) = \alpha + \beta$, $\tilde{p}_1(\alpha, \beta) = \alpha$, $\tilde{p}_2(\alpha, \beta) = \beta$ for $\alpha$, $\beta \in \pi_n$.

**Proposition 7.** The cohomology class $(\tilde{\psi}^* - \tilde{p}_1^* - \tilde{p}_2^*)k_{2n}^2(Y) \in H^{2n}(\pi_n \oplus \pi_n, n; \pi_{2n-1})$ defines a homomorphism $W : \pi_n \otimes \pi_n \rightarrow \pi_{2n-1}$ such that $W(\alpha \otimes \beta) = [\alpha, \beta]$ for $\alpha, \beta \in \pi_n$.

**Proof.** Consider the diagram,

\[
\begin{array}{ccc}
\text{Hom} \{ \pi_n \otimes \pi_n; \pi_{2n-1} \} & \xleftarrow{\theta_1} & H^{2n}(\pi_n \oplus \pi_n, n; \pi_{2n-1}) \\
\downarrow & & \downarrow \kappa^* \\
(\eta \otimes \eta)^* & H^{2n}(X \times X; \pi_{2n-1}) & \uparrow \iota_1^* \circ (\iota_2^*)^{-1} \\
\text{Hom} \{ H_n(Y) \otimes H_n(Y); \pi_{2n-1} \} & \xleftarrow{\theta_2} & H^{2n}(Y \times Y, Y \vee Y; \pi_{2n-1})
\end{array}
\]

The space $X$ is of type $(\pi_n, n)$, so that the natural chain maps from the cell complex of $X$ into the singular complex of $X$ and thence into $K(\pi_n, n)$ induces a chain map, $\kappa$, from the cell complex of $X \times X$ into $K(\pi_n \oplus \pi_n, n)$. This last induces the isomorphism $\kappa^*$. The homomorphisms $\tilde{\psi}$, $\tilde{p}_1$, $\tilde{p}_2$ are algebraic analogues of $\psi$, $p_1$, $p_2$. In particular, $\kappa^*(\tilde{\psi}^* - \tilde{p}_1^* - \tilde{p}_2^*)k_{2n}^2(Y) = (\psi^* - p_1^* - p_2^*)k$. The homomorphisms $i_1^*$, $i_2^*$.
are as defined in Theorem 5; \( \eta : \pi_n \rightarrow H_n(Y) \) is the Hurewicz isomorphism and \((\eta \otimes \eta)^*\) is the induced isomorphism of the Hom groups. The Kunneth formula yields the homomorphisms \( \theta_1, \theta_2; \theta_3 \) is an isomorphism. Note that \( \theta_1 \circ (\kappa^*)^{-1} \circ \iota_1^* \circ (\iota_2^*)^{-1} \circ \theta_2^* \circ ((\eta \otimes \eta)^*)^{-1} \) is the identity automorphism of \( \text{Hom} \{ \pi_n \otimes \pi_n; \pi_{n-1} \} \) onto itself.

Recall that \( d_n \in H^n(Y; \pi_n) \) is the basic cohomology class of \( Y \). Thus the image of \( d_n \) in \( \text{Hom} \{ H_n(Y); \pi_n \} \) is \( \eta^{-1} \) and \( \theta_1(W \circ (\eta^{-1} \otimes \eta^{-1})) = d_n \times d_n = \gamma \). We now have,

\[
W = \theta_1 \circ (\kappa^*)^{-1} \circ \iota_1^* \circ (\iota_2^*)^{-1} \circ \theta_2^* \circ ((\eta \otimes \eta)^*)^{-1}(W)
= \theta_1 \circ (\kappa^*)^{-1} \circ \iota_1^* \circ (\iota_2^*)^{-1}(\gamma)
= \theta_1 \circ (\psi^* - \rho_1^* - \rho_2^*)^2 k^n(Y).
\]

**Proposition 8.** Suppose \( Y \) is a CW-complex with only two nonvanishing homotopy groups, \( \pi_n = \pi_n(Y) \) and \( \pi_m = \pi_m(Y) \). Then there is a space of loops \( \Omega \) having the same homotopy groups and \( k \)-invariant as \( Y \) if and only if \( k^{m+1}_n(Y) \) is the suspension of an element

\[
k^{m+2} \in H^{m+2}(\pi_n, n + 1; \pi_m).
\]

**Proof.** Let \( S \) be the suspension homomorphism and suppose \( k^{m+1}_n(Y) = S k^{m+2}_n \). Let \( W \) be a space such that \( \pi_i(W) = 0 \) \( (i \neq n+1, m+1) \), \( \pi_{n+1}(W) = \pi_n \), \( \pi_{m+1}(W) = \pi_m \), \( k^{m+2}(W) = k^{m+2} \). Then \( \Omega \) is the space of loops on \( W \) is the desired space. The converse is immediate.

J. C. Moore has demonstrated (unpublished) that if \( \alpha \in H^{m+1}(\pi, n; G) \) is primitive then \( \alpha \) is the suspension of an element of \( H^{m+2}(\pi, n + 1; G) \). Thus all \( H \)-spaces of the type under discussion are essentially spaces of loops.

3. Let \( X \) be a given 0-connected space, \( x_0 \in X \). We construct a CW-complex, \( K(X) \), whose 0-skeleton is a point \( k_0 \in K(X) \), and a map \( f(X) : K(X) \rightarrow X \) such that \( f(X) \) induces isomorphisms \( f(X)_* : \pi_i(K(X), k_0) \rightarrow \pi_i(X, x_0) \) for \( i \geq 0 \). This is done by specifying the \( n \)-skeleta \( K^n \) of \( K(X) \) and maps \( f_n : K^n \rightarrow X \).

The 0-skeleton consists of one cell \( E^0 = k_0 \). Suppose \( K^n \) and \( f_n : K^n \rightarrow X \) are constructed such that the induced homomorphism

\[
f_n^{(i)} : \pi_i(K^n, k_0) \rightarrow \pi_i(X, x_0)
\]

is an isomorphism for \( i < n \) and onto for \( i = n \). Let \( A_{n+1} \) be the kernel of \( f_n^{(n)} \) and \( B_{n+1} \subset \pi_{n+1}(X, x_0) \) a set of generators of \( \pi_{n+1}(X, x_0) \). Append cells \( E_{\alpha}^{n+1}(\alpha \in A_{n+1}) \), so that if \( e_{\alpha}^{n+1} \) is the characteristic map of \( E_{\alpha}^{n+1} \) then \( (e_{\alpha}^{n+1} I_{n+1}) \subseteq \alpha \), and cells \( E_{\beta}^{n+1}(\beta \in B_{n+1}) \) with \( e_{\beta}^{n+1} I_{n+1} = k_0 \). The map \( f_n \) can be extended over cells \( E_{\alpha}^{n+1}(\alpha \in A_{n+1}) \) since

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$f_n|e^{n+1}_a(I^{n+1})$ is null-homotopic; $f_{n+1}|E^{n+1}_\beta (\beta \in B_{n+1})$ is determined by $f_{n+1}\circ e^{n+1}_\beta \in \beta$. Then

$$f^{(i)}_{n+1} : \pi_i(K^{n+1}, k_0) \to \pi_i(X, x_0)$$

is an isomorphism for $i < n+1$ and onto for $i = n+1$.

The complex $K(X)$ is $\bigcup_{n=0}^{\infty} K^n$, with the topology: $C$ is closed in $K(X)$ if and only if $C \cap K^n$ is closed in $K^n$ for each $n$. The map $f(X) : K(X) \to X$ is given by $(f(X)|K^n) = f_n$. Note that this is a modification of a construction by J. H. C. Whitehead [7].

**Proposition 9.** If $\pi_n, \pi_n$ are abelian groups ($n < m$) and $k \in H^{m+1}(\pi_n, n; \pi_m)$ then there is a CW-complex $K$ such that $\pi_i(K) = 0$ for $i \neq n, m$, $\pi_n(K) = \pi_n$, $\pi_m(K) = \pi_m$, $k^m = k$.

**Proof.** Let $E$ be the space of paths in the Eilenberg-MacLane space $K(\pi_m, m+1)$ terminating in some point $y_0 \in K(\pi_m, m+1)$ with fibre map $p_1 : E \to K(\pi_m, m+1)$ and fibre $K(\pi_m, m)$. If $d \in H^{m+1}(\pi_n, m+1; \pi_m)$ is the basic cohomology class, then there is a map $f : K(\pi_n, n) \to K(\pi_m, m+1)$ such that $f^*(d) = k \in H^{m+1}(\pi_n, n; \pi_m)$. Note that $K(\pi_n, m+1), K(\pi_n, n)$ may be chosen to be CW-complexes and $f$ cellular. The map $f$ induces a space $X$ and maps $p_2, F$ such that the diagram

$$\begin{array}{ccc}
X & \xrightarrow{F} & E \\
p_2 \downarrow & & \downarrow p_1 \\
K(\pi_n, n) & \to & K(\pi_m, m+1)
\end{array}$$

is commutative and $X$ is a fibre space over $K(\pi_n, n)$ with fibre map $p_2$ and fibre $K(\pi_m, m)$. From the homotopy sequence of the fibre map $p_2$ we see that $\pi_i(X) = 0$ for $i \neq n, m$, $\pi_n(X) = \pi_n$, $\pi_m(X) = \pi_m$.

We know that there is a map, $j$, of the $m$-skeleton of $K(\pi_n, n)$ into $X$ such that $p_2 \circ j$ is the identity, and that the obstruction to extending $j$ is $k^{m+1}_n$. If $E^{m+1}$ is an $(m+1)$-cell of $K(\pi_n, n)$ then its characteristic map $e^{m+1}$ (considered as a null-homotopy of $(e^{m+1} \mid \hat{I}^{m+1})$) can be lifted to a map $g : \hat{I}^{m+1} \times I \to X$ with $(g|\hat{I}^{m+1} \times 0) = j \circ (e^{m+1} \mid \hat{I}^{m+1})$ and $g' = (g|\hat{I}^{m+1} \times 1) : \hat{I}^{m+1} \to K(\pi_m, m)$. If $\partial$ is the boundary homomorphism of the homotopy sequence of $p_1$ and $F' = (F|K(\pi_m, m))$ then

$$\partial^{-1}F'\{g'\} = \{f \circ e^{m+1}\} \in \pi_{m+1}(K(\pi_m, m+1)).$$

But $f^*d(E^{m+1}) = \{f \circ e^{m+1}\}$. Thus there is an isomorphism of $H^{m+1}(\pi_n, n; \pi_m(X))$ onto $H^{m+1}(\pi_n, n; \pi_{m+1}(K(\pi_m, m+1)))$ carrying
$k^m+1$ into $k=f*d$. The desired CW-complex is then obtained as in the beginning of this section.

This proof is the obvious generalization of one given by Thom [5].

**Bibliography**


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