ON H-SPACES WITH TWO NONTRIVIAL HOMOTOPY GROUPS

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Introduction. Although several topologists (e.g. H. Hopf and A. Borel) have found necessary algebraic conditions for a space to admit an H-space structure, very little has been done towards obtaining sufficient conditions. The author believes that the present paper contains essentially the first result in the latter direction.

Let \( Y \) be a topological space with \( y_0 \in Y \), \( Y \cup Y = Y \times y_0 \cup y_0 \times Y \subset Y \times Y \). If \( \phi: Y \cup Y \to Y \) is the map given by \( \phi(y, y_0) = (y_0, y) = y \), then the problem of finding an H-space structure on \( Y \) may be expressed as the problem of extending \( \phi \) to a map \( \phi': Y \times Y \to Y \). It is found that if \( Y \) is a 1-connected, locally finite CW-complex [3], the obstructions to extending \( \phi \) may be expressed in terms of Postnikov invariants [4] and partial extensions of \( \phi \). If \( Y \) has only two nonzero homotopy groups then there is at most one nontrivial obstruction. This will be zero if and only if the Eilenberg-MacLane \( k \)-invariant of \( Y \) is primitive.

The relation between the existence of an H-structure and the vanishing of the J. H. C. Whitehead bracket products is investigated. This leads to a description of the lowest-dimensional bracket products on spaces whose first two nontrivial homotopy groups are in dimensions \( n \) and \( 2n - 1 \) (\( n > 1 \)).

1. This section presents much of the notation to be used, some discussion preliminary to the main result of the paper and an example of a space which is not an H-space, and yet has trivial bracket products.

If \( f_i: X_i \to Y_i \) (\( i = 1, 2 \)) are maps then the map \( f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2 \) is defined by \( f_1 \times f_2(x_1, x_2) = (f_1(x_1), f_2(x_2)) \) for \( x_i \in X_i \). If \( g_1: Y_1 \to Z_1 \), then \( g_1 \circ f_1: X_1 \to Z_1 \) is the map given by \( g_1 \circ f_1(x) = g_1(f_1(x)) \) for \( x \in X \), \( I = [0, 1] \) is the closed unit interval; \( I^m = I \times \cdots \times I \) (\( m \)-factors) is the unit \( m \)-cube; \( I^m \) is the usual boundary set of \( I^m \).

The discussion is restricted to locally finite CW-complexes, for if \( Y \) is a locally finite CW-complex then \( Y \times Y \) is a CW-complex whose (closed) \( m \)-cells may be taken to be of the form \( E^m = E^p \times E^q \) (\( p + q = m \); \( E^p, E^q \) are, respectively, \( p \)-, and \( q \)-cells of \( Y \)) [6]. The characteristic map of \( E^m \) is \( e^m = e^p \times e^q \); \( I^m \to E^m \) where \( e^p, e^q \) are character-
istic maps of $E^p, E^q$ and $I^m$ is identified with $I^p \times I^q$. Given cochains $c^p \in C^p(Y; G_1), c^q \in C^q(Y; G_2)$ and a pairing $\xi : G_1 \otimes G_2 \to G$ we form a cochain $c^p \times c^q \in C^{p+q}(Y \times Y; G)$ whose value on a cell $E^p \times E^q$ is $c^p \times c^q(E^p \times E^q) = \xi(c^p(E^p) \otimes c^q(E^q))$ and which is zero elsewhere. If $c^p, c^q$ are cocycles then $c^p \times c^q$ is, and the corresponding cohomology classes are $[c^p], [c^q], [c^p] \times [c^q] = [c^p \times c^q]$. If $X$ is a CW-complex, its $m$-skeleton will be designated by $X^m$. The map $\phi : Y \to Y$, presented in the introduction, will be called the folding map of $Y$. If $f : A \to B$ is a map, its homotopy class will be designated by $[f]$.

If the folding map has been extended over $Y \cup (Y \times Y)^m$ then the obstruction to extending over the $(m + 1)$-skeleton is in the cohomology group $H^{m+1}(Y \times Y, Y \vee Y; \pi_m(Y))$.

In case $Y$ is $(n - 1)$-connected, $(Y \times Y, Y \vee Y)$ will be $(2n - 1)$-connected.

**Proposition 1.** The obstruction in dimension $2n$ to extending the folding map is

$$d^n \times d^n \in H^{2n}(Y \times Y, Y \vee Y; \pi_{2n-1}(Y))$$

where $d^n \in H^n(Y, Y; \pi_n(Y))$ is the characteristic class for $Y$ and the pairing of $\pi_n(Y) \otimes \pi_n(Y)$ into $\pi_{2n-1}(Y)$ is the J. H. C. Whitehead bracket product.

**Proof.** We may replace $Y$ by a space whose $(n - 1)$-skeleton is a point (see §3). Then the $(2n - 1)$-skeleton of $Y \times Y$ is contained in $Y \vee Y$ so that the obstruction cocycle to extending $\phi$ is given by:

$$c^{2n}(E^n_1 \times E^n_2) = \{ \phi \circ (e^n_1 \times e^n_2) \mid I^{2n} \}$$

where $E^n_1, E^n_2$ are cells of $Y$ and $e^n_1, e^n_2$ their characteristic maps. Note that

$$\{ \phi \circ (e^n_1 \times e^n_2) \mid I^{2n} \} = \{ \{ e^n_1 \}, \{ e^n_2 \} \}$$

with $\{ e^n_1 \}, \{ e^n_2 \}$ regarded as elements of $\pi_n(Y)$ [1]. But $d_n$ is the class of the cocycle $c^n$ given by

$$c^n(E^n) = \{ e^n \}.$$

Thus $c^{2n} = c^n \times c^n$.

**Corollary 2.** $c^{2n} = 0$ if and only if $[\alpha, \beta] = 0$ for $\alpha, \beta \in \pi_n(Y)$.

**Proof.** Suppose $c^{2n} = 0$. Since $H_n(Y) \approx \pi_n(Y)$ the characteristic maps of $n$-cells generate $\pi_n(Y)$. Thus $[\alpha, \beta] = 0$ when $\alpha, \beta$ are in this set of generators; hence $[\alpha', \beta'] = 0$ for all $\alpha', \beta' \in \pi_n(Y)$. The converse is trivial.
On the other hand, if $Y$ is an $H$-space then all bracket products vanish. This raises the question: if $Y$ is a CW-complex and $[\pi_p(Y), \pi_q(Y)] = 0$ for $p, q > 0$, is $Y$ an $H$-space? The answer is negative, and we present a counter-example:

We construct a CW-complex, $K$, by specifying its $m$-skeleta, $K^m$. Let $n > 2$ be an integer and $p$ an odd prime. The $2n$-skeleton of $K$ is taken to be the $2n$-skeleton of a CW-complex which is an Eilenberg-MacLane space of type $(Z_p, n)$, where $Z_p$ is the integers mod $p$. There are no cells in dimension $2n + 1$ ($K^{2n+1} = K^{2n}$) and in higher dimensions, cells are appended so that $\pi_i(K) = 0$ for $i > 2n$ [7].

This creates a space whose homotopy groups are trivial except in dimensions $n$ and $2n$. Thus all bracket products vanish. Also note that $H^n(K; Z_p) \approx H^{n+1}(K; Z_p) \approx Z_p$ since the cohomology groups in these dimensions are those of a space of type $(Z_p, n)$. The $(2n+1)$-cohomology group is zero, since there are no $(2n+1)$-cells.

Now, $K$ is not an $H$-space, for if it were, its cohomology ring would be a Hopf algebra and the cup product of an element of $H^n(K; Z_p)$ with an element of $H^{n+1}(K; Z_p)$ would be nonzero, contradicting $H^{2n+1}(K; Z_p) = 0$ [2].

2. The main result. Let $Y$ be a 1-connected CW-complex, and suppose that the folding map has been extended to $\psi: Y \vee Y \cup (Y \times Y)^m \to Y$. Let $X$ be a CW-complex consisting of $Y$ united with $i$-cells, $E_i$, $(i > m)$ such that $\pi_i(X) = 0$ for $i \geq m$. Note that below dimension $m$, the inclusion map induces isomorphisms of the homotopy groups of $X$ and $Y$.

**Proposition 3.** $X$ is an $H$-space.

**Proof.** The $m$-skeleta of $X$ and $Y$ are the same, so that $(X \times X)^m = (Y \times Y)^m$. Thus $\phi$ provides an extension $\psi': X \vee X \cup (X \times X)^m \to X$. But $H^{i+1}(X \times X, X \vee X; \pi_i(X)) = 0$ for $i > m$, whence all obstructions to extending $\psi'$ vanish. One such extension, let us call it $\psi: X \times X \to X$, is chosen for the structure map of $X$.

The condition $\pi_i(X) = 0$ for $i \geq m$ also implies that any two extensions of $\psi'$ will be homotopic.

In the diagram below, $i_1^*, \ldots, i_5^*$ are homomorphisms induced by the appropriate inclusion maps: $\psi^*, \psi_i^*$ are induced by $\psi$. The coefficient group for each of these cohomology groups is $\pi_m(Y)$. This symbol has been omitted to save space. It is well known that $i_1^*$ sends $H^{m+1}(X \times X, X \vee X; \pi_m(Y))$ isomorphically onto a direct summand of $H^{m+1}(X \times X; \pi_m(Y))$. This direct sum decomposition induces the homomorphism $\rho$. The composition $\rho \circ i_1^*$ is the identity automorphism of $H^{m+1}(X \times X, X \vee X; \pi_m(Y))$, and the kernel of $\rho$ is essen-
tially $H^{m+1}(X \vee X; \pi_m(Y))$. The diagram is commutative.

$$
\begin{array}{c}
\xymatrix{
H^{m+1}(X \vee X) \ar[r]^{i_{2*}} \ar[d]_{i_1*} & H^{m+1}(X \times X, X \vee X) \ar[d]_{\rho} & \ar[l]^{i_2*} \ar[r]^{i_1*} & H^{m+1}(X \times X, X \vee X) \ar[d]_{i_4*} \ar[l]^{i_3*} & H^{m+1}(Y \times Y, Y \vee Y) \ar[d]_{i_4*} \ar[l]^{i_3*} \ar[r]^{i_1*} & H^{m+1}(X, Y) \ar[l]^{i_5*} & \}
\end{array}
$$

Let $k' \in H^{m+1}(X, Y; \pi_m(Y))$ be the first obstruction to retracting $X$ onto $Y$, and $k = i_2^* k' \in H^{m+1}(X; \pi_m(Y))$. The maps $p_1, p_2 : X \times X \to X$ are the projections $p_i(x_1, x_2) = x_i$ for $x_i \in X$, $i = 1, 2$.

**Proposition 4.** Let $\gamma \in H^{m+1}(Y \times Y, Y \vee Y; \pi_m(Y))$ be the class of the obstruction to extending $\phi$ to $(Y \times Y)^{m+1} \cup Y \vee Y$. Then

$$
i_2^* \rho(\psi^* - p_1^* - p_2^*) k = \gamma.
$$

**Proof.** The cohomology class $\psi_1^* k'$ is the first obstruction to extending $(\psi_1 | Y \vee Y \cup (X \times X)^m) = (\phi | Y \vee Y \cup (Y \times Y)^m)$ to $X \times X$ and hence $i_4^* \psi_1^* k' = \gamma$. On the other hand, $i_3^* \psi_1^* k' = \psi^* i_3^* k' = \psi^* k$, and so $i_2^* \rho \psi^* k = i_2^* \rho i_3^* \psi_1^* k' = i_2^* \psi^* k = \gamma$. Finally, $\rho \circ (p_1^* + p_2^*)$ is the trivial homomorphism, whence $i_2^* \rho (\psi^* - p_1^* - p_2^*) k = i_2^* \rho \psi^* k = \gamma$.

If $W$ is an $H$-space with $\psi, \rho_1, \rho_2 : W \times W \to W$ the structure map and the two projections respectively, then a cohomology class, $u$, is called *primitive* whenever $(\psi^* - p_1^* - p_2^*) u = 0$.

**Theorem 5.** The obstruction class, $\gamma$, vanishes if and only if $k$ is primitive.

**Proof.** We already have $i_2^* \rho (\psi^* - p_1^* - p_2^*) k = \gamma$ so that if $k$ is primitive, $\gamma = 0$. To obtain the converse, we first note that $i_2^*$ is an isomorphism since the inclusion map of $Y$ in $X$ induces isomorphisms $H^i(X) \cong H^i(Y)$ for $i \leq m$. But the image of $(\psi^* - p_1^* - p_2^*) k$ in $H^{m+1} \cdot (X \vee X; \pi_m(Y))$ is zero, since $(\psi \mid X \vee X)^* = (p_1 \mid X \vee X)^* + (p_2 \mid X \vee X)^*$. Thus $(\psi^* - p_1^* - p_2^*) k \in i_2^* H^{m+1}(X \times X, X \vee X; \pi_m(Y))$ from which it follows that $\rho (\psi^* - p_1^* - p_2^*) k = 0$ implies $(\psi^* - p_1^* - p_2^*) k = 0$. This completes the proof.

Note that $k$ is essentially the $(m+1)$-Postnikov invariant of $Y$. The theorem fails to provide a decisive victory over the problem of characterizing $H$-spaces which are CW-complexes, inasmuch as it depends upon choosing a particular extension, $\psi$, in each dimension. However, for sufficiently simple spaces the problem can be solved:

**Theorem 6.** Suppose $Y$ is a CW-complex which has only two non-trivial homotopy groups, $\pi_n(Y)$ and $\pi_m(Y)$, with $1 < n < m$. Then $Y$ is
an H-space if and only if the Eilenberg-MacLane k-invariant of Y is primitive.

Proof. The only nontrivial obstruction to extending the folding
map is in \( H^{m+1}(Y \times Y, Y \vee Y; \pi_m(Y)) \). Thus the space \( X \) is of type
\((\pi_n(Y), n)\) and \( k \) may be identified with the k-invariant, \( k_n^{m+1}(Y) \). The
result now follows from Theorem 4.

It may now be seen that the example of §1 was obtained by con-
structing a space with nonprimitive k-invariant.

In §3 it is shown that given abelian groups \( \pi_n, \pi_m \) and an element
\( k \in H^{m+1}(\pi_n, n; \pi_m) \), there is a space \( Y \) (which may be taken to be a
CW-complex) such that \( \pi_i(Y) = 0 \) for \( i \neq n, m \), \( \pi_n(Y) = \pi_n \), \( \pi_m(Y) = \pi_m \)
and \( k_n^{m+1}(Y) = k \). Any two such CW-complexes are of the same homo-
topy type. This observation and Theorem 5 give a classification of
CW-complexes which admit H-structures and have only two non-
vanishing homotopy groups.

Theorem 5 and Proposition 1 (§1) may be combined to yield a
result about the Whitehead bracket product. Suppose \( \pi_i(Y) = 0 \) for
\( 0 \leq i < n \) and \( n < i < 2n - 1 \). Let \( \pi_n = \pi_n(Y) \) and \( \pi_{2n-1} = \pi_{2n-1}(Y) \);
\( \pi_n \otimes \pi_n, \pi_n \oplus \pi_n \) designate respectively the tensor product and the
direct sum of \( \pi_n \) with itself. Three homomorphisms, \( \psi, \tilde{p}_1, \tilde{p}_2 : \pi_n \otimes \pi_n \rightarrow \pi_n \) are defined by
\( \psi(\alpha, \beta) = \alpha + \beta \), \( \tilde{p}_1(\alpha, \beta) = \alpha \), \( \tilde{p}_2(\alpha, \beta) = \beta \) for \( \alpha, \beta \in \pi_n \).

Proposition 7. The cohomology class \( (\psi^* - \tilde{p}_1^* - \tilde{p}_2^*)k_n^{2n}(Y) \in H^{2n}(\pi_n \otimes \pi_n, n; \pi_{2n-1}) \) defines a homomorphism \( W : \pi_n \otimes \pi_n \rightarrow \pi_{2n-1} \) such that
\( W(\alpha \otimes \beta) = [\alpha, \beta] \) for \( \alpha, \beta \in \pi_n \).

Proof. Consider the diagram,

\[
\begin{array}{ccc}
\text{Hom} \{ \pi_n \otimes \pi_n; \pi_{2n-1} \} & \xleftarrow{\theta_1} & H^{2n}(\pi_n \oplus \pi_n, n; \pi_{2n-1}) \\
\uparrow & & \downarrow \kappa^* \\
(\eta \otimes \eta)^* & & H^{2n}(X \times X; \pi_{2n-1}) \\
\text{Hom} \{ H_n(Y) \otimes H_n(Y); \pi_{2n-1} \} & \xleftarrow{\theta_2} & H^{2n}(Y \times Y, Y \vee Y; \pi_{2n-1}) \\
\end{array}
\]

The space \( X \) is of type \((\pi_n, n)\), so that the natural chain maps from
the cell complex of \( X \) into the singular complex of \( X \) and thence into
\( K(\pi_n, n) \) induces a chain map, \( \kappa \), from the cell complex of \( X \times X \) into
\( K(\pi_n \oplus \pi_n, n) \). This last induces the isomorphism \( \kappa^* \). The hom-
omorphisms \( \psi, \tilde{p}_1, \tilde{p}_2 \) are algebraic analogues of \( \psi, \tilde{p}_1, \tilde{p}_2 \). In particular,
\( \kappa^*(\psi^* - \tilde{p}_1^* - \tilde{p}_2^*)k_n^{2n}(Y) = (\psi^* - \tilde{p}_1^* - \tilde{p}_2^*)k \). The homomorphisms \( i_1^*, i_2^* \).
are as defined in Theorem 5; \( \eta : \pi_n \to H_n(Y) \) is the Hurewicz isomorphism and \((\eta \otimes \eta)^*\) is the induced isomorphism of the Hom groups. The Kunneth formula yields the homomorphisms \( \theta_1, \theta_2; \theta_2 \) is an isomorphism. Note that \( \theta_1 \circ (\kappa^*)^{-1} \circ (i_1^*)^{-1} \circ (\eta \otimes \eta)^{-1} \) is the identity automorphism of Hom \( \{ \pi_n \otimes \pi_n; \pi_{n-1} \} \) onto itself.

Recall that \( d_n \in H^n(Y; \pi_n) \) is the basic cohomology class of \( Y \). Thus the image of \( d_n \) in Hom \( \{ H^n(Y); \pi_n \} \) is \( \eta^{-1} \) and \( \theta_1(W \circ (\eta^{-1} \otimes \eta^{-1})) = d_n \times d_n = \gamma \). We now have,

\[
W = \theta_1 \circ (\kappa^*)^{-1} \circ i_1^* \circ (i_2^*)^{-1} \circ (\eta \otimes \eta)^{-1}(W)
= \theta_1 \circ (\kappa^*)^{-1} \circ i_1^* \circ (i_2^*)^{-1} (\gamma)
= \theta_1 \circ (\gamma^* - p_1^* - p_2^*)k_n^2(Y).
\]

**Proposition 8.** Suppose \( Y \) is a CW-complex with only two nonvanishing homotopy groups, \( \pi_n = \pi_n(Y) \) and \( \pi_m = \pi_m(Y) \). Then there is a space of loops \( \Omega \) having the same homotopy groups and \( k \)-invariant as \( Y \) if and only if \( k^{m+1}_n(Y) \) is the suspension of an element \( \kappa_{m+2} \in H^{m+2}(\pi, n + 1; \pi_m) \).

**Proof.** Let \( S \) be the suspension homomorphism and suppose \( k^{m+1}_n(Y) = Sk^{m+2}_n \). Let \( W \) be a space such that \( \pi_i(W) = 0 \) \((i \neq n+1, m+1)\), \( \pi_{n+1}(W) = \pi_n \), \( \pi_{m+1}(W) = \pi_m \), \( k^{m+2}_{n+1}(W) = k^{m+2}_n \). Then \( \Omega \) is the space of loops on \( W \) is the desired space. The converse is immediate.

J. C. Moore has demonstrated (unpublished) that if \( \alpha \in H^{m+1}(\pi, n; G) \) is primitive then \( \alpha \) is the suspension of an element of \( H^{m+2}(\pi, n+1; G) \). Thus all \( H \)-spaces of the type under discussion are essentially spaces of loops.

3. Let \( X \) be a given 0-connected space, \( x_0 \in X \). We construct a CW-complex, \( K(X) \), whose 0-skeleton is a point \( k_0 \in K(X) \), and a map \( f(X) : K(X) \to X \) such that \( f(X) \) induces isomorphisms \( f(X)^*) : \pi_i(K(X), k_0) \to \pi_i(X, x_0) \) for \( i \geq 0 \). This is done by specifying the \( n \)-skeleta \( K^n \) of \( K(X) \) and maps \( f_n : K^n \to X \).

The 0-skeleton consists of one cell \( E^0 = k_0 \). Suppose \( K^n \) and \( f_n : K^n \to X \) are constructed such that the induced homomorphism

\[
f_n^{\langle i \rangle} : \pi_i(K^n, k_0) \to \pi_i(X, x_0)
\]

is an isomorphism for \( i < n \) and onto for \( i = n \). Let \( A_{n+1} \) be the kernel of \( f_n^{(n)} \) and \( B_{n+1} \subset \pi_{n+1}(X, x_0) \) a set of generators of \( \pi_{n+1}(X, x_0) \). Append cells \( E_{\alpha}^{n+1} (\alpha \in A_{n+1}) \), so that if \( e_{\alpha}^{n+1} \) is the characteristic map of \( E_{\alpha}^{n+1} \) then \( (e_{\alpha}^{n+1})^* \in \alpha \), and cells \( E_{\beta}^{n+1} (\beta \in B_{n+1}) \) with \( e_{\beta}^{n+1} (\beta^{n+1}) = k_0 \). The map \( f_n \) can be extended over cells \( E_{\alpha}^{n+1} (\alpha \in A_{n+1}) \) since
$f_n|_{e_\alpha^{n+1}}(i^{n+1})$ is null-homotopic; $f_{n+1}|E^{n+1}_\beta (\beta \subseteq B_{n+1})$ is determined by $f_{n+1} \circ e_\beta^{n+1} \subseteq \beta$. Then

$$f^{(i)}_{n+1}: \pi_i(K^{n+1}, k_0) \to \pi_i(X, x_0)$$

is an isomorphism for $i < n+1$ and onto for $i = n+1$.

The complex $K(X)$ is $\bigcup_{n=0}^\infty K^n$, with the topology: $C$ is closed in $K(X)$ if and only if $C \cap K^n$ is closed in $K^n$ for each $n$. The map $f(X): K(X) \to X$ is given by $(f(X)|K^n) = f_n$. Note that this is a modification of a construction by J. H. C. Whitehead [7].

**Proposition 9.** If $\pi_n, \pi_m$ are abelian groups ($n < m$) and $k \in \pi^{m+1}(\pi_n, n; \pi_m)$ then there is a CW-complex $K$ such that $\pi_i(K) = 0$ for $i \neq n, m$, $\pi_n(K) = \pi_n$, $\pi_m(K) = \pi_m$, $k^m = k$.

**Proof.** Let $E$ be the space of paths in the Eilenberg-MacLane space $K(\pi_m, m+1)$ terminating in some point $y_0 \in K(\pi_m, m+1)$ with fibre map $p_1: E \to K(\pi_m, m+1)$ and fibre $K(\pi_m, m)$. If $d \in H^{m+1}(\pi_m, m+1; \pi_m)$ is the basic cohomology class, then there is a map $f: K(\pi_n, n) \to K(\pi_m, m+1)$ such that $f^*(d) = k \in H^{m+1}(\pi_n, n; \pi_m)$. Note that $K(\pi_m, m+1), K(\pi_n, n)$ may be chosen to be CW-complexes and $f$ cellular. The map $f$ induces a space $X$ and maps $p_2, F$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{F} & E \\
p_2 & \downarrow & \downarrow p_1 \\
K(\pi_n, n) & \to & K(\pi_m, m+1)
\end{array}
$$

is commutative and $X$ is a fibre space over $K(\pi_n, n)$ with fibre map $p_2$ and fibre $K(\pi_m, m)$. From the homotopy sequence of the fibre map $p_2$ we see that $\pi_i(X) = 0$ for $i \neq n, m$, $\pi_n(X) = \pi_n$, $\pi_m(X) = \pi_m$.

We know that there is a map, $j$, of the $m$-skeleton of $K(\pi_n, n)$ into $X$ such that $p_2 \circ j$ is the identity, and that the obstruction to extending $j$ is $k^{m+1}_n$. If $E^{m+1}$ is an $(m+1)$-cell of $K(\pi_n, n)$ then its characteristic map $e^{m+1}$ (considered as a null-homotopy of $(e^{m+1}|i^{m+1})$) can be lifted to a map $g: i^{m+1} \times I \to X$ with $(g|i^{m+1} \times 0) = j \circ (e^{m+1}|i^{m+1})$ and $g' = (g|i^{m+1} \times 1): i^{m+1} \to K(\pi_m, m)$. If $\partial$ is the boundary homomorphism of the homotopy sequence of $p_1$ and $F' = (F| K(\pi_m, m))$ then

$$\partial^{-1}F'(g') = \{f \circ e^{m+1}\} \subseteq \pi_{m+1}(K(\pi_m, m+1)).$$

But $f^*d(E^{m+1}) = \{f \circ e^{m+1}\}$. Thus there is an isomorphism of $H^{m+1}(\pi_n, n; \pi_m(X))$ onto $H^{m+1}(\pi_n, n; \pi_{m+1}(K(\pi_m, m+1)))$ carrying
$k^{m+1}_n$ into $k = f^*d$. The desired CW-complex is then obtained as in the beginning of this section.

This proof is the obvious generalization of one given by Thom [5].

**Bibliography**