NOTE ON SUMS OF FOUR AND SIX SQUARES

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1. Bailey [1] showed that Ramanujan's identity

$$\sum_{m=0}^{\infty} p(5m + 4) = 5 \prod_{n=1}^{\infty} \frac{(1 - x^{5n})^5}{(1 - x^n)^6}$$

can be derived from the identity

$$\sum_{n=-\infty}^{\infty} \left\{ \frac{xq^n}{(1 - xq^n)^2} - \frac{yq^n}{(1 - yq^n)^2} \right\} = \frac{(x - y)(1 - xy)}{(1 - x)^2(1 - y)^2}$$

$$\prod_{n=1}^{\infty} \frac{(1 - xq^n)(1 - x^{-1}y^{-1}q^n)(1 - xy^{-1}q^n)(1 - x^{-1}yq^n)(1 - q^n)^4}{(1 - xq^n)^2(1 - x^{-1}q^n)^2(1 - yq^n)^2(1 - y^{-1}q^n)^2}$$

which is equivalent to the familiar formula

$$\varphi(u) = \varphi(v) = - \frac{\sigma(u + v)\sigma(u + v)}{\sigma^2(u)\sigma^2(v)} .$$

Similarly the formula

$$1 + a^{-1} \frac{(1 - a)^3}{1 + a} \sum_{n=1}^{\infty} \frac{n^2 q^{2n}}{1 - q^{2n}} (a^n - a^{-n})$$

$$= \prod_{n=1}^{\infty} \frac{(1 - q^{2n}a^2)(1 - q^{2n}a^{-2})(1 - q^{2n})^6}{(1 - q^{2n}a)^4(1 - q^{2n}a^{-1})^4} ,$$

which is equivalent to

$$\varphi'(u) = - \frac{\sigma(2u)}{\sigma^4(u)} ,$$

can be used to prove various results involving partition functions. Dobbie [3] recently constructed simple direct proofs of (1) and (2) that require no knowledge of elliptic functions; incidentally (2) can be derived from (1) by dividing by $$x-y$$ and then letting $$y \to x$$.

The writer [2] showed that by means of (2) one can give a very concise proof of the familiar formula for the number of representations of an integer as a sum of eight squares or of eight odd squares. In the present note we obtain the formulas for four and six squares in a similar manner (see for example [6, p. 307]).

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2. We recall the formulas (see for example [5, p. 282])

\[
\theta_0(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^n^2 = \prod_{1}^{\infty} \frac{(1 - q^n)^2}{1 - q^{2n}},
\]

(3)

\[
\theta_2(q) = 2 \sum_{1}^{\infty} q^{(2n-1)^2/4} = 2q^{1/4} \prod_{1}^{\infty} \frac{(1 - q^{4n})^2}{1 - q^{2n}},
\]

(4)

\[
\theta_3(q) = \theta_0(-q).
\]

(5)

It follows from (3) and (4) that

\[
\theta_0(q)\theta_3(q) = \theta_0(q^2), \quad \theta_3(q) = 2\theta_2(q^2)\theta_3(q^2).
\]

(6)

For the case of six squares we shall in addition require

\[
\theta_3(q) = \theta_0(q) + \theta_3(q),
\]

(7)

which incidentally is proved in §3 below.

We define \(r_k(n), r'_k(n)\) by means of

\[
\theta_k(q) = \sum_{n=0}^{\infty} r_k(n)q^n, \quad \theta_k(q^4) = \sum_{n=1}^{\infty} r'_k(n)q^n.
\]

(8)

3. In (1) replace \(q\) by \(q^2\) and then put \(y = -x = q\). The left hand side of (1) becomes

\[
\sum_{n=-\infty}^{\infty} \left\{ \frac{q^{2n+1}}{(1 - q^{2n+1})^2} + \frac{q^{2n+1}}{(1 + q^{2n+1})^2} \right\} = 2 \sum_{n=0}^{\infty} \left\{ \frac{q^{2n+1}}{(1 - q^{2n+1})^2} + \frac{q^{2n+1}}{(1 + q^{2n+1})^2} \right\}.
\]

The right hand side of (1) becomes

\[
4q \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^4}{(1 - q^{4n+2})^4} = 4q \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^8}{(1 - q^{2n})^4} = \frac{1}{4} \theta_2(q).
\]

Hence we have the identity

\[
\sum_{n=0}^{\infty} (8n + 4)q^{2n+1} = \theta_3(q) = 8 \sum_{n=0}^{\infty} \left\{ \frac{q^{2n+1}}{(1 - q^{2n+1})^2} + \frac{q^{2n+1}}{(1 + q^{2n+1})^2} \right\}
\]

(9)

\[
= 16 \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} (2r + 1)q^{(2n+1)(2r+1)},
\]

which is equivalent to the known results on sums of four odd squares.

In (1) let us now put \(x = i\) and \(y = -i\). The left hand side of (1) becomes
\[
\sum_{n=-\infty}^{\infty} \left\{ \frac{iq^n}{(1 - iq^n)^2} + \frac{q^n}{(1 + q^n)^2} \right\} = -\frac{1}{4} + \sum_{n=1}^{\infty} \left\{ \frac{iq^n}{(1 - iq^n)^2} - \frac{iq^n}{(1 + iq^n)^2} + \frac{2q^n}{(1 + q^n)^2} \right\} = -\frac{1}{4} + \sum_{n=1}^{\infty} \left\{ -\frac{4q^{2n}}{(1 + q^2n)^2} + \frac{2q^n}{(1 + q^n)^2} \right\} = -\frac{1}{4} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{(1 + q^n)^2}.
\]

The right hand side of (1) becomes
\[
-\frac{1}{4} \prod_{n=1}^{\infty} \frac{(1 + q^n)^2(1 - iq^n)^2(1 - q^n)^4}{(1 - iq^n)^2(1 + iq^n)^2(1 + q^n)^4} = -\frac{1}{4} \prod_{n=1}^{\infty} \frac{(1 - q^n)^8}{(1 - q^2n)^4} = -\frac{1}{4} \theta_0^4(q).
\]

Hence we have the identities
\[
(10) \quad \theta_0^4(q) = 1 + 8 \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{(1 + q^n)^2} = 1 + 8 \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} (-1)^{n+r+1} rq^{nr},
\]
\[
\sum_{n=0}^{\infty} r_4(n)q^n = \theta_3^4(q) = \theta_0^4(-q) = 1 + 8 \sum_{n=1}^{\infty} \frac{q^n}{(1 + (-q)^n)^2}
\]
\[
(11) \quad = 1 + 8 \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} (-1)^{(n-1)(r-1)} rq^{nr},
\]

the latter of which is equivalent to the known results on sums of four squares.

From (9), (10) and (11) we see that
\[
\theta_3^4(q) = \theta_0^4(q) + \theta_2^4(q),
\]
which is (7) above. The writer is indebted to the referee for this observation.

4. Turning next to (2), we take \(a = i\). This yields
\[
1 - 4 \sum_{m=1}^{\infty} \left( \frac{-4}{m} \right) \frac{m^2 q^{2m}}{1 - q^{2m}} - \prod_{1}^{\infty} \frac{(1 - q^{2n})^4(1 - q^{4n})^8}{(1 - q^{2n})^4},
\]
where \((-4/m)\) is the Jacobi symbol. The right member is equal to
\[
\prod_{1}^{\infty} \frac{(1 - q^{2n})^4(1 - q^{4n})^8}{(1 - q^{2n})^4(1 - q^{4n})^8} = \theta_0^2(q) \theta_0^2(q) = \theta_0^4(q) \theta_3^2(q),
\]
where we have used (3) and (6). We have therefore

\[ 1 - 4 \sum_{m=1}^{\infty} \left( \frac{-4}{m} \right) \frac{m^2 q^{2m}}{1 - q^{2m}} = \theta_0^4(q) \theta_2^2(q^2). \]

Now take \( a = qi \) in (2). We find that the right member becomes

\[ 2 \frac{(1 - qi)^4}{1 + q^2} \prod_{i=1}^{\infty} \frac{(1 - q^{2n})^8(1 - q^{6n})^4}{(1 - q^{6n})^8} = \frac{(1 - qi)^3}{1 + qi} \frac{\theta_0(q^2) \theta_2(q^2)}{2q}; \]

the left member is equal to

\[
\begin{align*}
1 + \frac{1}{q} \frac{(1 - qi)^3}{1 + qi} \sum_{i=1}^{\infty} \frac{n^2 q^{2n}}{1 - q^{2n}} i^{n-1}(q^n - (-1)^n q^{-n}) \\
= \frac{(1 - qi)^3}{q(1 + qi)} \left\{ q(1 + qi) + \sum_{i=1}^{\infty} \frac{n^2 q^{2n}}{1 - q^{2n}} i^{n-1}(q^n - (-1)^n q^{-n}) \right\} \\
= 2 \frac{(1 - qi)^3}{q(1 + qi)} \sum_{i=1}^{\infty} \sum_{i=1}^{\infty} (-1)^{n-1}(2n - 1)^2 q^{(2r-1)(2n-1)},
\end{align*}
\]

on expanding and combining. Thus we get

\[ 4 \sum_{i=1}^{\infty} \sum_{i=1}^{\infty} (-1)^{n-1}(2n - 1)^2 q^{(2r-1)(2n-1)} = \theta_0(q) \theta_2(q^2). \]

If we divide by \( q \), replace \( q^2 \) by \(-q^2\), we find that (13) becomes

\[ 4 \sum_{i=1}^{\infty} \sum_{i=1}^{\infty} (-1)^{r-1}(2n - 1)^2 q^{(2r-1)(2n-1)} = \theta_2(q) \theta_2(q^2). \]

Again, if we take \( a = q^{1/2} \) in (2) and then replace \( q \) by \( q^4 \), we get without much difficulty

\[
\begin{align*}
1 + q^{-2} \frac{(1 - q^2)^2}{1 + q^2} \sum_{i=1}^{\infty} \frac{n^2 q^{6n}}{1 + q^{4n}} = \frac{(1 - q^2)^8}{1 + q^2} \prod_{i=1}^{\infty} \frac{(1 - q^{8n})^4(1 - q^{4n})^8}{(1 - q^{4n})^8(1 - q^{8n})^4}, \\
64q^2 \frac{1 + q^2}{(1 - q^2)^3} + 64 \sum_{i=1}^{\infty} \frac{n^2 q^{6n}}{1 + q^{4n}} = \theta_2(q) \theta_2(q) + \theta_2(q^2) \theta_2(q^2);
\end{align*}
\]

hence by the second of (6)

\[ 16q^2 \frac{1 + q^2}{(1 - q^2)^3} + 16 \sum_{i=1}^{\infty} \frac{n^2 q^{6n}}{1 + q^{4n}} = \theta_2(q) \theta_2(q^2). \]

If we subtract (13) from (14) and use (7), it is evident that
(16) \( \theta_6^2(q^2) = 4 \sum_{1}^{\infty} \sum_{1}^{\infty} \frac{1}{(-1)^{r-1} - (-1)^{s-1}} (2s - 1)^2 q^{(2r-1)(2s-1)}. \)

Define

\[ E_2(n) = \sum_{d|n} \left( \frac{-4}{d} \right) d^2, \quad E_4(n) = \sum_{d|n} \left( \frac{-4}{d} \right) d^2; \]

then the right member of (16) becomes

\[ 4 \sum_{m \text{ odd}} q^m \{ E_4^2(m) - E_2(m) \}. \]

This evidently implies

(17) \( r_6^2(2m) = 4 \{ E_4^2(m) - E_2(m) \} \quad (m \text{ odd}). \)

On the other hand, addition of (12) and (15) gives after some simplification

\[ \theta_6^2(q^2) = 1 + 16 \sum_{1}^{\infty} \sum_{1}^{\infty} \left( \frac{-4}{r} \right) n^2 q^{2nr} - 4 \sum_{1}^{\infty} \sum_{1}^{\infty} \left( \frac{-4}{n} \right) n^2 q^{2nr}, \]

which implies

(18) \( r_6(n) = 16E_4^2(n) - 4E_2(n). \)

The formulas (17) and (18) are the well-known results of Jacobi on six squares; the notation is that of Glaisher [4].

We remark that (14) and (15) imply results on the number of representations in the forms

\[ 4(x_1^2 + x_2^2 + x_3^2 + x_4^2) + u_1 + u_2, \quad 4(x_1^2 + x_2^2) + u_1 + u_2 + u_3 + u_4, \]

where the \( u_i \) are odd, \( x_i \) arbitrary.

References


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