NOTE ON SUMS OF FOUR AND SIX SQUARES

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1. Bailey [1] showed that Ramanujan's identity

\[ \sum_{m=0}^{\infty} p(5m + 4) = 5 \prod_{n=1}^{\infty} \frac{(1 - x^{5n})^5}{(1 - x^n)^6} \]

can be derived from the identity

\[ \sum_{n=-\infty}^{\infty} \left\{ \frac{x q^n}{(1 - x q^n)^2} - \frac{y q^n}{(1 - y q^n)^2} \right\} = \frac{(x - y)(1 - xy)}{(1 - x)^2(1 - y)^2} \]

\[ \prod_{n=1}^{\infty} \frac{(1 - xy q^n)(1 - x^{-1} y^{-1} q^n)(1 - xy^{-1} q^n)(1 - x^{-1} y q^n)(1 - q^n)^4}{(1 - x q^n)^2(1 - x^{-1} q^n)^2(1 - y q^n)^2(1 - y^{-1} q^n)^2} \]

which is equivalent to the familiar formula

\[ \varphi(u) - \varphi(v) = - \frac{\sigma(u + v)\sigma(u + v)}{\sigma^2(u)\sigma^2(v)}. \]

Similarly the formula

\[ 1 + a^{-1} \frac{(1 - a)^3}{1 + a} \sum_{n=1}^{\infty} \frac{n^2 q^{2n}}{1 - q^{2n}} (a^n - a^{-n}) \]

\[ = \prod_{n=1}^{\infty} \frac{(1 - q^{2n} a^2)(1 - q^{2n} a^{-2})(1 - q^{2n})^6}{(1 - q^{2n} a)^4(1 - q^{2n} a^{-1})^4}, \]

which is equivalent to

\[ \varphi'(u) = - \frac{\sigma(2u)}{\sigma^4(u)}, \]

can be used to prove various results involving partition functions. Dobbie [3] recently constructed simple direct proofs of (1) and (2) that require no knowledge of elliptic functions; incidentally (2) can be derived from (1) by dividing by \(x-y\) and then letting \(y \rightarrow x\).

The writer [2] showed that by means of (2) one can give a very concise proof of the familiar formula for the number of representations of an integer as a sum of eight squares or of eight odd squares. In the present note we obtain the formulas for four and six squares in a similar manner (see for example [6, p. 307]).

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2. We recall the formulas (see for example [5, p. 282])

\[ \theta_0(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^n = \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{1 - q^{2n}}, \]

\[ \theta_2(q) = 2 \sum_{n=1}^{\infty} q^{(2n-1)^2/4} = 2q^{1/4} \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^2}{1 - q^{2n}}, \]

\[ \theta_3(q) = \theta_0(-q). \]

It follows from (3) and (4) that

\[ \theta_0(q)\theta_2(q) = \theta_0(q^2), \quad \theta_3^2(q) = 2\theta_2(q^2)\theta_3(q^2). \]

For the case of six squares we shall in addition require

\[ \theta_3(q) = \theta_0(q) + \theta_2(q), \]

which incidentally is proved in §3 below.

We define \( r_k(n), r_k'(n) \) by means of

\[ \theta_3^k(q) = \sum_{n=0}^{\infty} r_k(n) q^n, \quad \theta_3^k(q^4) = \sum_{n=0}^{\infty} r_k'(n) q^n. \]

3. In (1) replace \( q \) by \( q^4 \) and then put \( y = -x = q \). The left hand side of (1) becomes

\[
\sum_{n=-\infty}^{\infty} \left\{ \frac{q^{2n+1}}{(1 - q^{2n+1})^2} + \frac{q^{2n+1}}{(1 + q^{2n+1})^2} \right\} = 2 \sum_{n=0}^{\infty} \left\{ \frac{q^{2n+1}}{(1 - q^{2n+1})^2} + \frac{q^{2n+1}}{(1 + q^{2n+1})^2} \right\}.
\]

The right hand side of (1) becomes

\[
4q \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^4}{(1 - q^{4n-2})^4} = 4q \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^8}{(1 - q^{2n})^4} = \frac{1}{4} \theta_2(q).
\]

Hence we have the identity

\[
\sum_{n=0}^{\infty} r_4'(8n + 4)q^{2n+1} = 8 \sum_{n=0}^{\infty} \left\{ \frac{q^{2n+1}}{(1 - q^{2n+1})^2} + \frac{q^{2n+1}}{(1 - q^{2n+1})^2} \right\} = 16 \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} (2r + 1)q^{(2n+1)(2r+1)},
\]

which is equivalent to the known results on sums of four odd squares.

In (1) let us now put \( x = i \) and \( y = -i \). The left hand side of (1) becomes
\[
\sum_{n=0}^{\infty} \left\{ \frac{iq^n}{(1 - iq^n)^2} + \frac{q^n}{(1 + q^n)^2} \right\}
= -\frac{1}{4} + \sum_{n=1}^{\infty} \left\{ \frac{iq^n}{(1 - iq^n)^2} - \frac{iq^n}{(1 + iq^n)^2} + \frac{2q^n}{(1 + q^n)^2} \right\}
= -\frac{1}{4} + \sum_{n=1}^{\infty} \left\{ -\frac{4q^{2n}}{(1 + q^{2n})^2} + \frac{2q^n}{(1 + q^n)^2} \right\}
= -\frac{1}{4} - 2\sum_{n=1}^{\infty} (-1)^n \frac{q^n}{(1 + q^n)^2}.
\]

The right hand side of (1) becomes
\[
\frac{1}{4} \prod_{n=1}^{\infty} \frac{(1 - iq^n)^2(1 - iq^n)^2(1 - q^n)^4}{(1 - iq^n)^2(1 + iq^n)^2(1 + q^n)^4}
= -\frac{1}{4} \prod_{n=1}^{\infty} \frac{(1 - q^n)^8}{(1 - q^{2n})^4} = -\frac{1}{4} \theta_0(q).
\]

Hence we have the identities
\[
\theta_0^4(q) = 1 + 8 \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{(1 + q^n)^2} = 1 + 8 \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} (-1)^{n+r+1} r q^{nr},
\]
\[
\sum_{n=0}^{\infty} r_4(n) q^n = \theta_3^4(q) = \theta_0^4(-q) = 1 + 8 \sum_{n=1}^{\infty} \frac{q^n}{(1 + (-q)^n)^2}
= 1 + 8 \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} (-1)^{n-1} (r-1) r q^{nr},
\]
the latter of which is equivalent to the known results on sums of four squares.

From (9), (10) and (11) we see that
\[
\theta_3^4(q) = \theta_0^4(q) + \theta_2^4(q),
\]
which is (7) above. The writer is indebted to the referee for this observation.

4. Turning next to (2), we take \( a = i \). This yields
\[
1 - 4 \sum_{m=1}^{\infty} \left( -\frac{4}{m} \right) \frac{m^2 q^{2m}}{1 - q^{2m}} = \prod_{1}^{\infty} \frac{(1 - q^{2n})^4(1 - q^{4n})^6}{(1 - q^{8n})^4},
\]
where \( (-4/m) \) is the Jacobi symbol. The right member is equal to
\[
\prod_{1}^{\infty} \frac{(1 - q^{2n})^4(1 - q^{4n})^8}{(1 - q^{4n})^2(1 - q^{8n})^4} = \theta_0(q) \theta_3(q) = \theta_0^2(q) \theta_3^2(q),
\]
where we have used (3) and (6). We have therefore

\[
1 - 4 \sum_{m=1}^{\infty} \left( \frac{-4}{m} \right) \frac{m^2 q^{2m}}{1 - q^{2m}} = \theta_0^4(q) \theta_2^2(q^2).
\]

Now take \(a = qi\) in (2). We find that the right member becomes

\[
2 \frac{(1 - qi)^4}{1 + q^2} \prod_{i=1}^{\infty} \frac{(1 - q^{2n})^8(1 - q^{6n})^4}{(1 - q^{4n})^8} = \frac{(1 - qi)^3}{1 + qi} \frac{\theta_0^4(q^2) \theta_2^2(q^2)}{2q};
\]

the left member is equal to

\[
1 + \frac{(1 - qi)^3}{q(1 + qi)} \sum_{i=1}^{\infty} \frac{n^2 q^{2n}}{1 - q^{2n}} i^{n-1}(q^n - (-1)^n q^{-n})
\]

\[
= \frac{(1 - qi)^3}{q(1 + qi)} \left\{ \frac{q(1 + qi)}{(1 - qi)^3} + \sum_{i=1}^{\infty} \frac{n^2 q^{2n}}{1 - q^{2n}} i^{n-1}(q^n - (-1)^n q^{-n}) \right\}
\]

\[
= 2 \frac{(1 - qi)^3}{q(1 + qi)} \sum_{i=1}^{\infty} \sum_{i=1}^{\infty} (-1)^{n-1}(2n - 1)^2 q^{(2n-1)(2n-1)},
\]

on expanding and combining. Thus we get

\[
4 \sum_{i=1}^{\infty} \sum_{i=1}^{\infty} (-1)^{n-1}(2n - 1)^2 q^{(2n-1)(2n-1)} = \theta_0^4(q) \theta_2^2(q^2).
\]

If we divide by \(q\), replace \(q^2\) by \(-q^2\), we find that (13) becomes

\[
4 \sum_{i=1}^{\infty} \sum_{i=1}^{\infty} (-1)^{n-1}(2n - 1)^2 q^{(2n-1)(2n-1)} = \theta_3^4(q) \theta_2^2(q^2).
\]

Again, if we take \(a = q^{1/2}\) in (2) and then replace \(q\) by \(q^4\), we get without much difficulty

\[
1 + \frac{q^2}{1 + q^2} \sum_{i=1}^{\infty} \frac{n^2 q^{6n}}{1 + q^{4n}} = \frac{(1 - q^2)^8}{1 + q^2} \prod_{i=1}^{\infty} \frac{(1 - q^{8n})^8(1 - q^{4n})^8}{(1 - q^{4n})^8(1 - q^{8n})^8},
\]

\[
64 q^2 \frac{1 + q^2}{(1 - q^2)^3} + 64 \sum_{i=1}^{\infty} \frac{n^2 q^{6n}}{1 + q^{4n}} = \theta_2^2(q^2) \theta_4^2(q^2);
\]

hence by the second of (6)

\[
16 q^2 \frac{1 + q^2}{(1 - q^2)^3} + 16 \sum_{i=1}^{\infty} \frac{n^2 q^{6n}}{1 + q^{4n}} = \theta_2^2(q^2) \theta_4^2(q^2).
\]

If we subtract (13) from (14) and use (7), it is evident that
Define
\[ E_2(n) = \sum_{d|n} \left( -\frac{4}{d} \right) d^2, \quad E_4(n) = \sum_{d|n} \left( -\frac{4}{d} \right) d^2; \]
then the right member of (16) becomes
\[ 4 \sum_{m \text{ odd}} q^{m} \{ E'_2(m) - E_2(m) \}. \]
This evidently implies
\[ r_6(2m) = 4 \{ E'_2(m) - E_2(m) \} \quad (m \text{ odd}). \]
On the other hand, addition of (12) and (15) gives after some simplification
\[ \theta_6(q^2) = 1 + 16 \sum_{r=1}^{\infty} \sum_{1}^{\infty} \left( -\frac{4}{r} \right) n^2 q^{2nr} - 4 \sum_{r=1}^{\infty} \sum_{1}^{\infty} \left( \frac{4}{n} \right) n^2 q^{2nr}, \]
which implies
\[ r_6(n) = 16 E'_2(n) - 4 E_2(n). \]
The formulas (17) and (18) are the well-known results of Jacobi on six squares; the notation is that of Glaisher [4].

We remark that (14) and (15) imply results on the number of representations in the forms
\[ 4(x_1 + x_2 + x_3 + x_4) + u_1 + u_2, \quad 4(x_1^2 + x_2^2) + u_1 + u_2 + u_3 + u_4, \]
where the \( u_i \) are odd, \( x_i \) arbitrary.

References


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