ON A GENERALIZATION OF THE NOTION OF $H^*$-ALGEBRA

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1. Introduction. An $H^*$-algebra of W. Ambrose [1] has the property that the orthogonal complement of an ideal is an ideal of the same kind. The present work is an attempt to characterize an $H^*$-algebra in terms of this property. For this purpose it is necessary to generalize the concept of $H^*$-algebra by introducing so-called two-sided $H^*$-algebras. We assume merely that there are two involutions in the algebra: right $x \mapsto x^r$ and left $x \mapsto x^l$ such that $(yx, z) = (y, zx^r)$ and $(xy, z) = (y, x^l z)$. It is possible to characterize two-sided $H^*$-algebra in terms of the above relation imposed on ideals by making some additional assumptions on the ideal annihilators. Since every simple two-sided $H^*$-algebra is an $H^*$-algebra with the same topology we have also found a new characterization of a proper $H^*$-algebra.

The notation is adopted essentially from [1; 6] and [3] but unlike Ambrose we do not require that ideals be closed. We shall make a distinction between a minimal ideal and a minimal closed ideal. Also it is understood that a proper ideal is not dense in whole algebra and that an idempotent is a nonzero element.

2. Complemented and right complemented algebra. First structure theorem.

Definition 1. Let $A$ be a Banach algebra which is a Hilbert space. We shall call $A$ a right complemented algebra (r. c. algebra) if it has the property that the orthogonal complement of every right ideal is again a right ideal. Similarly we define a left complemented algebra (l. c. algebra). We shall call an algebra complemented if it is at the same time r. c. and l. c.

As an example of a right complemented algebra one can take a right $H^*$-algebra introduced by M. F. Smiley [6].

Example 1. Let $\alpha$ be a positive norm-increasing bounded operator on a Hilbert space $H$ and let $A$ be the algebra of operators of the Hilbert Schmidt type on $H$. Then $A$ is a right $H^*$-algebra (hence a r. c. algebra) in the scalar product $(a, b) = [aa, b]$, where $[ , ]$ denote a trace scalar product of the Hilbert Schmidt operators: $[a, b] = \text{tr} (a^*b)$.

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1 Most of the results in this paper were taken from the author's doctoral thesis, Harvard University, 1954, written under the direction of Professor L. H. Loomis.
Definition 2. A right complemented algebra $A$ will be called proper if $r(A) = \{x \in A \mid Ax = (0)\} = (0)$.

Lemma 1. The orthogonal complement $I^p$ of a two-sided ideal $I$ in a proper right complemented algebra $A$ is a two-sided ideal and is identical with both the left and the right annihilator of $I$.

Proof. We know already that $I^p$ is a right ideal. Since $I^pI \subseteq I^p \cap I = (0)$, $I^p$ is contained in $l(I)$ which is a two-sided ideal. Hence it is sufficient to prove that $I^p = l(I)$. Consider $I_1 = I \cap l(I)$, which is also a two-sided ideal. It is readily verified that $I_1 \subseteq r(A) = (0)$. Thus $l(I) = I^p$. It follows at once that $I^p = r(I)$.

It is easy to see that a semi-simple r. c. algebra is proper and that every r. c. algebra is a direct sum of its radical and a semi-simple r. c. algebra. Also it is always true for a proper r. c. algebra $A$ that $l(A) = (0)$.

We shall say that an element $x \in A$ is left self-adjoint if $(xy, z) = (y, xz)$ holds for every $y, z \in A$. An element $e$ will be called a left projection if it is idempotent and left self-adjoint.

Lemma 2. Let $R$ be a proper closed regular right ideal in an r. c. algebra $A$ and let $e$ be its relative identity such that $e \in R^p$. Then $e$ is a left projection.

Proof. Since $e$ is a relative identity we have $ex - x \in R$ for every $x \in A$. It follows that $eR = 0$ and $eR^p = R^p$; if $x \in R$, then $ex \in R$ and, since $ex \in R^p$ also, we have $ex = 0$; if $x \in R^p$, then $ex - x \in R^p \cap R$ and hence $ex - x = 0$, $ex = x$. In particular $e^2 = e$. Also $e$ is left self-adjoint for if we consider $x, y \in A$ and write $x = x_1 + x_2$, $y = y_1 + y_2$ with $x_1, y_1 \in R^p$, $x_2, y_2 \in R$, we have:

$$(ex, y) = (ex_1 + ex_2, y_1 + y_2) = (x_1, y_1 + y_2) = (x_1, y_1).$$

$$(x, ey) = (x_1 + x_2, ey_1 + ey_2) = (x_1 + x_2, y_1) = (x_1, y_1).$$

Lemma 3. Every semi-simple r. c. algebra $A$ contains a left projection.

Proof. Let $x$ be an element in $A$ which does not have a right quasi-inverse. Let $R$ be the closure of the regular right ideal $\{xy + y \mid y \in A\}$. Then $-x$ is a relative identity of $R$. We write $-x = e + u$ with $e \in R^p$, $u \in R$; then one can easily check that $e$ also is a relative identity of $R$. Hence $e$ is a left projection.

Now we define the double orthogonality and the primitivity of a projection as in [1]. Also as in [1] we show that every left projection can be expressed as a finite sum of doubly orthogonal primitive left projections, and the closed right ideal $eA$, where $e$ is a projection, is a
minimal closed ideal if and only if \( e \) is primitive. From the first part of the last statement it follows that every semi-simple r. c. algebra contains a primitive left projection.

**Theorem 1.** Every semi-simple r. c. algebra \( A \) is a direct sum of simple r. c. algebras each of which is a two-sided ideal in \( A \).

**Proof.** Let \( e \) be a primitive left projection in \( A \) and let \( I \) be the smallest closed two-sided ideal containing \( e \). It is easy to see that \( I \) is a minimal closed two-sided ideal. Then \( I^p \) is also a semi-simple r. c. algebra. Using Zorn's lemma we complete the proof.

This is the first structure (Wedderburn) theorem for r. c. algebras.

3. **Two-sided \( * \)-algebras.**

**Definition 3.** A Banach algebra \( A \) is called a two-sided \( * \)-algebra if \( A \) is a Hilbert space and if for every \( a \in A \) there are elements \( a^l \) and \( a^r \) in \( A \) such that \((ab, c) = (b, a^l c)\) and \((ba, c) = (b, c a^r)\) hold for every \( b, c \in A \).

**Theorem 2.** Every proper right \( * \)-algebra \( A \) is a two-sided \( * \)-algebra.

**Proof.** Let \( x \in A \) and let \( N_1 \) be the linear space spanned by \( x \). Let \( M_1 = N_1^\perp, M_2 = M_1^\perp = \{ y \in A \mid y^r \in M_1 \} \) and \( N_2 = M_2^\perp \). Then \( N_2 \) is one-dimensional. Take \( u \in N_2 \) so that \((x, x) = (x^r, u)\) (note that \((x^r, u) = 0\) for all \( u \in N_2 \) is impossible). We shall show that \( u = x^l \). Let \( y, z \in A \); then \( z y^r = \lambda x + v \) where \( \lambda \) is some complex number and \( v \in M_1 \). It follows that \( y z^r = \lambda x^r + v^r \) with \( v^r \in M_2 \). Then we have: \((x y, z) = (x, z y^r) = (x, \lambda x + v) = (x, \lambda x) = \lambda (x, x) = \lambda (x^r, u) = (\lambda x^r, u) = (\lambda x^r + v^r, u) = (y z^r, u) = (y, u z)\).

4. **Well-complemented algebra. Second structure theorem.** It turns out that in order to prove the second structure theorem for (right, left) complemented algebras it is necessary to introduce a new axiom. The algebras with the new axiom will be called well-complemented.

**Definition 4.** A semi-simple r. c. algebra \( A \) will be called right well-complemented (r. w. c.) if every proper right (left) ideal in \( A \) has a nonzero left (right) annihilator. A c. algebra will be called well-complemented (w. c.) if it is r. w. c.

A two-sided \( * \)-algebra furnishes an example of a w. c. algebra.

Now we prove a series of lemmas which will be used to prove the second structure theorem.

**Lemma 4.** Let \( L \) be a left ideal in a Banach algebra \( A \) such that every
member of $L$ has a right quasi-inverse. Then $L$ is contained in the radical of $A$.

**Proof.** The lemma follows from the fact that if $xy$ has a right quasi-inverse $u$ then $yx$ has also a right quasi-inverse $v = -(yx + yux)$.

**Lemma 5.** Every closed nonzero right ideal $R$ in an r. w. c. algebra $A$ contains a left projection.

**Proof.** Since $A$ is semi-simple $l(R)$ contains an element $x$ which does not have a right quasi-inverse. Consider the closed regular right ideal $R_1 = \text{closure of } \{xy + y | y \in A \}$ of which $-x$ is a relative identity. Since $xR_p = (0)$, we have $R_p \subseteq R_1$ and hence $R_1 \subseteq R$. We write $-x = e + u$ with $e \in R_p$, $u \in R_1$; then $e$ is again a relative identity of $R_1$. By Lemma 2 $e$ is a left projection.

**Lemma 6.** If $e$ is a primitive idempotent in an r. w. c. algebra $A$, then the right ideal $P = eA$ is a minimal right ideal.

**Proof.** We know already that $P$ is a minimal closed ideal. So it is sufficient to show that if $R$ is an ideal dense in $P$ then $e \in R$. In this case we can find $x \in R$ such that $x - e$ has a right quasi-inverse $y$. Then $xy + x - e = 0$ and hence $e \in R$.

Combining Lemma 6 with the technique used in [1] we prove:

**Lemma 7.** Let $\{e_i\}$ be a family of primitive idempotents in a simple algebra $A$. Let $A_{ij} = e_i A e_j$. Then:

(i) Each $A_{ii}$ is isomorphic to the complex field;
(ii) Each $A_{ij}$ is one-dimensional;
(iii) $A_{ij} A_{jk} = A_{ik}$.

We shall say that an element $x$ in a c. algebra $A$ has a left adjoint if there exists an element $x^l \in A$ such that $(xy, a) = (y, x^l z)$ holds for all $y, z \in A$.

**Lemma 8.** If $e$ is a primitive left projection in an r. w. c. algebra $A$ then every element in $Ae$ has a left adjoint.

**Proof.** Let $x \in Ae$. Consider the right ideal $R = xA = xeA$. We may assume that $ex \neq 0$ (otherwise we replace $x$ by $y = x + e$ and prove that $y$ has a left adjoint). Then since $exe = xe$ we have $eR = eA$ from which it follows that $R$ is closed. Hence $R$ contains a left projection $f = xz = xeze$, where $z$ is some element in $A$. Then $fe \neq 0$ (otherwise $0 = ef =exe = xeze$, since $ef$ is left adjoint of $fe$) and hence $fe = xze = \mu x$ or $x = 1/\mu fe$ from which follows that $x$ has a left adjoint $x^l = 1/\mu ef$.

Now we are in position to prove the second structure theorem.
Theorem 3. Every r. w. c. algebra is a two-sided $H^*$-algebra. In particular a simple r. w. c. algebra $A$ is isomorphic to the algebra described in Example 1.

Proof. We follow the technique used in [1] and [6]. Let $\{e_i\}$ be a maximal family of doubly orthogonal primitive left projections in $A$. Consider $R = \sum_{i \in J} e_i A$, where $J$ is the set of indices in $\{e_i\}$. Then $R$ is closed. If $R \neq A$ then $R^p$ contains a primitive left projection $e$. Then $(e; e, e) = (e, e; e) = 0$, $e_i e = 0$ for every $i \in J$, i.e., $e$ is doubly orthogonal to every $e_i$, which leads to a contradiction. Thus $R = A$. Consider $L = \sum_{i \in J} A e_i$ and suppose $L \neq A$. Then the right annihilator $r(L)$ is a nonzero right ideal. This simply means that there is an element $x \in A$ such that $e_i x = 0$ for all $e_i$. Then for any $y \in e_i A$ we have $(e_i y, x) = (y, e_i x) = 0$, i.e., $x$ is orthogonal to all $e_i A$, hence to whole $A$, which is a contradiction. Thus $L$ is dense in $A$. Let us use the notation $A_{ij} = e_i A e_j$; then $L = \sum_{i,j} A_{ij}$. It follows from Lemma 8 that every element in $A_{ij}$ has a left adjoint (in $A_{ji}$). So we choose the matrix units $e_{ij} \in A_{ij}$ such that $e_{ii} = e_i$ and $e_{ij}^* = e_{ji}$. We define the matrix $(\alpha_{ij})$ by setting $\alpha_{ij} = (e_{ki}, e_{kj})$. It is easy to see that $\alpha_{ij}$ does not depend upon $k$ and that the matrix $(\alpha_{ij})$ is self-adjoint. Any two elements in $L$ have the form $x = \sum_{i,j} x_{ij} e_{ij}$ and $y = \sum_{i,j} y_{ij} e_{ij}$, where $x_{ij}$ and $y_{ij}$ are suitable complex numbers, and the scalar product has the form:

$$(x, y) = \sum x_{ik} \alpha_{kj} y_{ij} = \text{tr}(x \alpha y^*),$$

where $x$, $y$ and $\alpha$ here stand for matrices $(x_{ij})$, $(y_{ij})$ and $(\alpha_{ij})$ respectively.

Now we shall show that $(\alpha_{ij})$ represents a bounded operator on $L^2(J)$. For this purpose let us consider the conjugate-linear mapping $T: x \rightarrow x^i$ restricted to $A e_1$, where $1$ is some fixed index in $J$. Since every element in $A e_1$ has a left adjoint $T$ is defined everywhere on $A e_1$; the range of $T$ is a subset of $e_1 A$ (in fact one can show that the range is entire $e_1 A$). The graph of $T$ is closed: if $\langle x_n, x'_n \rangle \rightarrow \langle x, u \rangle$, then $x_n \rightarrow x$ and $x'_n \rightarrow u$, and for every $y, z \in A (x_n y, z) \rightarrow (x y, z)$ and $(y, x'_n z) \rightarrow (y, u z)$ from which it follows that $x^i = u$, and so $\langle x, u \rangle$ also belongs to the graph of $T$. From the closed graph theorem it follows that $T$ is continuous. Thus there exists a positive number $M$ such that

$$(*) \quad (x^i, x^i) \leq M(x, x)$$

holds for all $x \in A e_1$.

Now there is a natural 1-1 correspondence between elements of $L^2(J)$ and $A e_1$, in which a member $x(i)$ of $L^2(J)$ corresponds to the element $x = \sum_{i \in J} x(i) e_i$ and $A e_1$. If $x(i)$ and $y(i)$ are finite sequences
in $L^2(J)$ then $x' = \sum_i \hat{x}(i)e_{1i}$, $y' = \sum_i \hat{y}(i)e_{1i}$ and $(x', y') = \sum_{i,j} \hat{x}(i)\alpha_{ij}\hat{y}(j)$. By the continuity of $T_{x', y'} = \sum_{i,j} \hat{x}(i)\alpha_{ij}\hat{y}(j)$ holds for all $x(i), y(i)$ in $L^2(J)$. From (*) we have then $\sum_{i,j} x(i)\alpha_{ij}\hat{y}(j) \leq a_{1i}M \sum_i |x(i)|^2$. Completing the proof as in [6] we show that $L$ is an algebra of the type described in Example 1. Since $L$ is complete we have $L = A$. So $A$ is a left and hence a two-sided $H^*$-algebra.

To conclude this section we construct a complemented algebra which is not well-complemented.

**Example 2.** Let $\tilde{a}$ be some positive norm-increasing unbounded operator with domain dense in some Hilbert space $H$. Let $A$ be the set of all operators $a$ on $H$ such that $a\tilde{a}$ is an operator of the Hilbert Schmidt type. Then $A$ is a complemented algebra in the scalar product $(a, b) = [a\tilde{a}, b\tilde{a}] = \text{tr}(a\tilde{a}(b\tilde{a}))$. It is easy to see that there are two dense subsets of elements in $A$, every element of one having the left adjoint $a^l = a^*$ and every element of the other having the right adjoint $a^r = \tilde{a}a^*\tilde{a}^{-2}$ in $A$. It remains to show that $A$ is not well-complemented. Let us denote by $\tilde{A}$ the algebra of all operators of the Hilbert Schmidt type on $H$, and let $e$ be some left projection in $A$. It is easy to show that the Hilbert space $H$ can be realized as $\tilde{A}e$ and so that if $x \in H$ corresponds to $x \in \tilde{A}e$ and $\alpha$ is any operator such that $\alpha(x)$ is defined then $\alpha(x) = \alpha x$. Since $\tilde{a}$ is unbounded there exists an $a \in \tilde{A}e = Ae$ such that $\tilde{a}a$ is not of the Hilbert Schmidt type. This means that $a$ does not have the left adjoint. Then from Lemma 8 it follows that $A$ is not well-complemented.

5. A special realization of a well-complemented algebra.

**Example 3.** Let $\lambda$ be a norm-decreasing linear transformation from a Hilbert space $H_2$ onto a Hilbert space $H_1$ which has a bounded inverse transformation $\mu$ from $H_1$ onto $H_2$. Let $A$ be the set of all Hilbert Schmidt operators $a$ from $H_1$ into $H_2$. Let us define the multiplication by $a \circ b = a\lambda b$. Then $A$ is a w. c. algebra in the scalar product $(a, b) = \text{tr}(ab^*)$. All the laws of an algebra are easily verified; $\|ab\| \leq \|a\| \|b\|$ follows from the fact that $\lambda$ is norm-decreasing. The right and the left adjoint of an element $a \in A$ are defined by $a^r = \mu a^*\lambda^*$ and $a^l = \lambda^*a^*\mu$.

It turns out that every simple w. c. algebra $A$ is of the above form. This is shown in the next theorem.

**Theorem 4.** Every simple w. c. algebra $A$ is of the form described in Example 3.

**Proof.** Let $\{e_i\}$ and $\{f_k\}$ be maximal families of doubly orthogonal primitive left and right projections in $A$ respectively. Then $A = \sum_{i,k} e_iAf_k$. We choose $e_{ij}$ and $f_{ki}$ as in Theorem 3, such that
Let \( J = \{ j \} \) and \( K = \{ k \} \) be the sets of indices in \( \{ e_j \} \) and \( \{ f_k \} \) respectively; let \( 1 \in J \) and \( 2 \in K \) be some fixed indices. Choose \( w_{12} \in e_1 A_f \) such that \( \| w_{12} \| = 1 \) and let \( w_{jk} = e_{j1} w_{12} f_{k2} \) for each \((j, k) \in J \times K\). Then \( w_{jk} \in e_j A_f \) and \( \| w_{jk} \| = 1 \). Since every \( e_j A_f \) is one-dimensional, \( w_{jk} \) constitute an orthonormal base for \( A \). Thus every \( x \in A \) has the form \( x = \sum_{j, k} x(j, k) w_{jk} \) and so the scalar product is of the form \( (x, y) = \sum_{j, k} x(j, k) y(j, k) \).

Now consider \( w_{ik} w_{j2} \); since \( w_{ik} w_{j2} \in e_1 A_f \) we have \( w_{ik} w_{j2} = \lambda_{kj} w_{12} \) for some complex \( \lambda_{kj} \). Multiplying both sides of the last equality with \( e_{i1} \) on the left and with \( f_{22} \) on the right we get \( w_{ik} w_{j1} = \lambda_{kj} w_{i1} \). Thus the multiplication has the form \( xy = \sum_{i, k} x(i, k) w_{ik} \sum_{j, l} y(j, l) w_{jl} = \sum_{i, l}( \sum_{k, j} x(i, k) \lambda_{kj} y(j, l) ) w_{il} \).

It remains to show that the matrix \( \lambda_{kj} \) regarded as an operator from \( H_2 = L^2(J) \) into \( H_1 = L^2(K) \) is norm-decreasing and has an inverse. Since \( w_{ik} \in e_i A \) we have \( w_{ik} = \sum_h \tau_{kh} e_{ih} \) for some matrix \( \{ \tau_{kh} \} \) which can be shown to be independent of \( i \); also \( e_{ih} = \sum_l \mu_{hi} w_{il} \) where \( \{ \mu_{hi} \} \) is independent of \( i \). It is easy to show that \( \{ \mu_{hi} \} \) is the inverse of \( \{ \tau_{hi} \} \). From the other hand \( \lambda_{kj} w_{il} = w_{ik} w_{j1} = \sum_h \tau_{kh} e_{ih} w_{j1} = \tau_{kj} e_{ij} w_{j1} = \tau_{kj} w_{il} \) which simply means that \( \lambda_{kj} = \tau_{kj} \) and that \( \{ \mu_{hi} \} \) is an inverse of \( \{ \lambda_{kj} \} \). In order to show that \( \{ \lambda_{kj} \} \) is norm-decreasing we consider \( \alpha_{ij} = (e_{i1}, e_{1j}) = (\sum_k \mu_{ik} w_{1k}, \sum_l \mu_{lj} w_{1l}) = \sum_k \mu_{ik} \bar{\mu}_{jk} \) which simply means that \( \alpha = \mu \mu^* \). From the fact that \( \alpha \) is positive and norm-increasing it is easy to derive that \( \mu \) is also norm-increasing. The rest of the proof follows immediately.

Thus the algebra of Example 3 is essentially a most general w. c. algebra. All w. c. algebras are obtained by considering all possible direct sums of the algebras of the form of that described in Example 3.

**Bibliography**

5. G. W. Mackey, *Commutative Banach algebra*, mimeographed notes prepared by A. Blair, Harvard University, 1952.