Let \( A \) be a Banach algebra with a Hilbert space norm (norm defined by a scalar product). We shall call \( A \) a right complemented algebra if it has the property that the orthogonal complement of a right ideal is again a right ideal. This notion was introduced in the author's doctoral thesis [5]. It was proved that under certain additional assumptions every right complemented algebra is left complemented. We shall prove this theorem for a general right complemented algebra. We shall also show that the most general simple right (left) complemented algebra is of the following form.

**Example.** Let \( \alpha \) be a (possibly unbounded) self-adjoint linear operator with domain dense in a Hilbert space \( H \) and the range being a subset of \( H \). Let \( A \) be the algebra of all linear operators \( a \) of the Hilbert Schmidt type on \( H \) such that \( |a\alpha| < \infty \), where \(| \cdot |\) is the trace norm of an operator: \( |a|^2 = \text{tr} (a^*a) \). Then \( A \) is a right (as well as left) complemented algebra in the scalar product \( (a, b) = [a\alpha, b\alpha] = \text{tr} (aa(ba)^*) \).

We shall use the following terminology (see [5]). A Banach algebra shall be called simple if it is semi-simple and has no proper two-sided ideals except those which are dense in whole algebra. We shall say that \( x' \) is the left adjoint of \( x \) if \( (xy, z) = (y, x'z) \) holds for all \( y, z \) in the algebra. A left projection \( e \) is a left self-adjoint (nonzero) idempotent; a primitive left projection is a left projection which cannot be written as a sum of two doubly orthogonal left projections (compare with W. Ambrose [1]). The orthogonal complement of an ideal \( I \) will be denoted by \( I^\perp \).

We have proved in [5] that every simple right complemented algebra has a primitive left projection. So we begin by proving:

**Theorem 1.** Let \( A \) be a simple right complemented algebra and let \( e \) be a primitive left projection in \( A \). Then every element in \( eA \) has a left adjoint.

**Proof.** Let \( a \in eA \); then \( ea = a \). We may assume that \( ae \neq 0 \) (otherwise we consider \( b = a + e \) for which \( be \neq 0 \)). Then \( a^2 = eaea = \lambda a \), i.e., \( a \) is a multiple of some idempotent \( f \). Consider the closed regular right ideal \( Q = \{ z - fz | z \in A \} \), \( f \) is a relative identity of \( Q \). We write
Theorem 2. The set of elements in a simple right complemented algebra $A$ having left adjoint is dense in $A$.

Proof. Let $F = \{e_i\}$ be the family of all primitive left projections in $A$. Let $R$ be the closed right ideal generalized by $F$, i.e., $R$ is the closure of the linear space spanned by all elements of the form $e_i x$, $e_i \in F$, $x \in A$. It follows from Lemma 1 that the set of elements in $R$ having a left adjoint is dense in $R$. It remains to show that $R = A$. Suppose $R \not= A$, then $R^p \not= (0)$. Let $a \in R^p$ be an element which does not have a right quasi-inverse. Consider the right regular ideal $Q = \text{closure of } \{ax + x | x \in A \}$ for which $-a$ is relative identity. We write $-a = e + u$ with $e \in Q^p$, $u \in Q$. Then it is easy to see that $e$ is a left projection (of course $e \not= 0$) such that $eu = 0$ (compare with [5, Lemma 2]). Thus $e \not= 0$ and hence $(ea, ea) = (ea, a) = 0$, $ea = 0$. But on the other hand $-ea = e(e + u) = e$, which is a contradiction. Thus $R = A$.

Corollary. Every semi-simple right complemented algebra $A$ is a left complemented algebra; the set of elements in $A$ having right adjoint is dense in $A$.

From now on we may refer to a semi-simple right complemented algebra simply as a "complemented algebra."

Now we proceed with the second part of our paper. Let $A$ be a simple complemented algebra and let $e$ be a primitive left projection in $A$. We consider the ideals $L = Ae$ and $R = eA$. Every element in $R$ has a left adjoint while $L$ has a dense subset of elements having left adjoint. We shall show that $A$ is a dense subalgebra of a suitably constructed $H^*$-algebra. It will be done by proving a series of lemmas in which $A$, $e$ (and hence $L$ and $R$) are fixed once and for all.

Lemma 1. If $x_1, x_2 \in L$ and $y_1, y_2 \in R$, then $(x_1 y_1, x_2 y_2) = \omega^{-2}(x_1, x_2) \cdot (y_1, y_2)$ where $\omega = \|e\|$.

Proof. Since $x_2 x_1 \in eAe$ we have $x_2^* x_1 = \lambda e$ for some complex $\lambda$. Since $eAe$ is isomorphic to the complex field [5, Lemma 7)]. Then $(x_1, x_2) = (x_1 x_2, e) = (\lambda e, e) = \lambda \|e\|^2 = \lambda \omega^2$ and $(x_1 y_1, x_2 y_2) = (x_2^* x_1 y_1, y_2) = (\lambda y_1, y_2) = \omega^{-2}(x_1, x_2) (y_1, y_2)$.

Corollary. If $x \in L$ and $y \in R$ then $\|xy\| = \omega^{-1} \|x\| \|y\|$.

Lemma 2. If $x \in R$ then $\|x^*\| \leq \omega \|x\|$.

Proof. If $x \in R$, then $xx^* = \lambda e$ for some positive $\lambda$ (we again use the
fact that $eAe$ is isomorphic to the complex field). So we have:
\[
\|x'||^2 = (x'e, x'e) = (xx', e) = \lambda(e, e) = \lambda\|e\| \cdot \|e\| = \|\lambda e\| \cdot \|e\|
\]
\[
= \|xx'\| \cdot \|e\| \leq \|x\| \cdot \|x'|| \cdot \|e\|
\]
or \[
\|x'|| \leq \|x\| \cdot \|e\| = \omega\|x\|.
\]

**Lemma 3.** If an element has the form $z = \sum_{i=1}^{n} x_iy_i$ with $x_i \in L$, $y_i \in R$, then $x_1, x_2, \ldots, x_n$ can be so chosen in $L$ that $(x_i, x_j) = 0$ for $i \neq j$; also $y_1, y_2, \ldots, y_n$ can be so chosen in $R$ that $(y_i^t, y_j^t) = 0$ for $i \neq j$.

**Proof.** The lemma is easily proved by induction.

Now consider $S = LR = AeA$. We define the function $[,]$ on $S \times S$ by setting
\[
[x_1x_1, y_2y_2] = \frac{1}{\omega^4} (x_1, x_2)(y_2, y_1)
\]
where $\omega = \|e\|$. (It is understood that $x_1, x_2 \in L$, $y_1, y_2 \in R$.)

**Lemma 4.** The function $[,]$ is independent of the choice of the primitive left projection $e$.

**Proof.** Let $e_1$ and $e_2$ be any two primitive left projections. Suppose $z_i = x_iy_i$, $i = 1, 2$, with $x_i \in Ae_1$ and $y_i \in e_1A$. Then $[x_1y_1, x_2y_2] = 1/\omega^4(x_1, x_2)(y_2, y_1)$, where $\omega = \|e_1\|$. We shall show that $z_i \in Ae_2A$ and that $[z_1, z_2] = [z_1, z_2]_2$, where $[,]_2$ is the above function defined with respect to $e_2$.

It can be easily shown that there are elements $e_{12}$ and $e_{21}$ in $A$ such that $e_{12} = e_{21}$, $e_{12}e_{21} = e_1$, $e_{12}e_{21} = e_2$, $e_{12}e_{21} = e_1$ and $e_{12}e_{21} = e_2$. Then $z_i = x_iy_i = x_i e_{12} e_{21} y_i$, and hence $z_i \in Ae_2A$. Also
\[
[z_1, z_2]_2 = \frac{1}{\|e_2\|^4} (x_1 e_{12}, x_2 e_{12})(y_2 e_{12}, y_1 e_{12})
\]
\[
= \frac{1}{(e_{21} e_{12}, e_{21} e_{12})^2} \frac{1}{\omega_1^2} \frac{1}{\omega_2^2} (x_1, x_2)(e_{12}, e_{12})(y_2, y_1)(e_{12}, e_{12})
\]
\[
= \frac{1}{(e_{12}, e_{12})^2} \frac{1}{\omega_1^4} (x_1, x_2)(y_2, y_1)(e_{12}, e_{12})^2
\]
\[
= \frac{1}{\omega_1^4} (x_1, x_2)(y_2, y_1) = [z_1, z_2]_1.
\]

**Lemma 5.** The function $[,]$ has the following properties:
(a) \([\lambda x, y] = \lambda [x, y]\)
(b) \([x, y] = \text{complex conjugate of } [y, x]\).
(c) \([x, x] \geq 0 \text{ and } [x, x] = 0 \text{ if and only if } x = 0\).
(d) \([\sum_{i=1}^{n} z_i, z] = \sum_{i=1}^{n} [z_i, z]\), provided \(z_i, z \in S\) and \(\sum_{i=1}^{n} z_i \in S\).

**Proof.** (a)–(c) are easily verified. We shall prove (d). Since \(z_i, z \in S\) we have \(z_i = x_i y_i, z = xy\) and also \(u = \sum_{i=1}^{n} z_i = vw\) with \(x_i, x, v, v \in L, y_i, y, w \in R\). Let us assume that \(z_1, z_2, \cdots, z_n, x\) are fixed while \(y\) is variable. We have: \((u, z) = (vw, xy) = \omega^{-2}(v, x)(w, y)\) or \((v, x)(w, y) = \omega^2 \sum_{i=1}^{n} (x_i y_i, xy) = \sum_{i=1}^{n} (x_i, x)(y_i, y)\). Now let us assume that \((v, x) \neq 0\). This can be done without loss of generality. Then we can write \((x_i, x) = \lambda_i(v, x)\) for some complex \(\lambda_i, i = 1, 2, \cdots, n\) and so we have:

\[
(v, x)(w, y) = \sum_{i=1}^{n} \lambda_i(v, x)(y_i, y) = (v, x) \sum_{i=1}^{n} (\lambda_i y_i, y)
\]

or \((w, y) = (\sum_{i=1}^{n} \lambda_i y_i, y)\). It can be written \((w - \sum_{i=1}^{n} \lambda_i y_i, y) = 0\), where \(y\) is an arbitrary element in \(R\). This simply means that \(w = \sum_{i=1}^{n} \lambda_i y_i\) (note that \(w, y \in R\)).

Now let us take \(y\) so that \(z = xy\). Then we have:

\[
[u, z] = [vw, xy] = \frac{1}{\omega^4} (v, x)(y^t, w^t) = \frac{1}{\omega^4} (v, x) \left( y^t, \sum_{i=1}^{n} \lambda_i y_i^t \right)
\]

\[
= \frac{1}{\omega^4} \sum_{i=1}^{n} \lambda_i(v, x)(y_i^t, y_i^t) = \frac{1}{\omega^4} \sum_{i=1}^{n} (x_i, x)(y_i^t, y_i^t)
\]

\[
= \sum_{i=1}^{n} [x_i, y_i, xy] = \sum_{i=1}^{n} [z_i, z].
\]

Now let \(I\) be the set of all finite sums of elements in \(S\), i.e., \(I\) is the set of all elements of the form \(\sum_{i=1}^{n} x_i y_i\) with \(x_i \in L, y_i \in R\). It is easy to see that \(I\) is a two-sided ideal dense in \(A\).

**Lemma 6.** The function \([\ , \ ]\) has a unique extension to \(I\), which has the properties of a scalar product.

**Proof.** If \(z = \sum_{i=1}^{n} z_i\) and \(u = \sum_{j=1}^{n} u_j\) with \(z_i \in S, u_j \in S\), then we define \([z, u] = \sum_{i,j} [u_i, z_j]\). It is easy to verify that \([\ , \ ]\) is a scalar product, using Lemma 5. The uniqueness of \([\ , \ ]\) follows from (d).

**Lemma 7.** If \(u, v \in I\) then \(|w| \leq |u| |v|\), where \(|\ |\) denotes the corresponding to \([\ , \ ]\) norm.

**Proof.** (a) We first take \(u, v \in S\), then \(u = x_1 y_1, v = x_2 y_2\) with \(x_i \in L, y_i \in R\). Then \(w = x_1 y_1 x_2 y_2 = \lambda x_1 y_2\), since \(y_1 x_2 = \lambda e\) for some \(\lambda\), and
\[ |uv| = |\lambda| \cdot |x_1y_2| = \frac{1}{\omega^2} |\lambda| \cdot \|x_1\| \cdot \|y_2\|. \]

But \( |\lambda|\omega^2 = |\lambda| (e, e) = |(y_1x_2, e)| = |(x_2, y_1')| \leq \|x_2\| \cdot \|y_1'\| \),

i.e., \( |\lambda| \leq \omega^{-2} \|x_2\| \cdot \|y_1'\| \).

Hence
\[ |uv| \leq \frac{1}{\omega^2} \|x_1\| \cdot \|y_1'\| \cdot \frac{1}{\omega^2} \|x_2\| \cdot \|y_2\| = |u| \cdot |v|. \]

(b) Suppose that \( u \in I \) and \( v \in S \); then \( u = \sum_{i=1}^n x_iy_i \). We may assume that \( (x_i, x_j) = 0 \) for \( i \neq j \). Then \( [x_iy_i, x_jy_j] = 0 \) and \( [x_iy_i, x_jy_j] = 0 \) and hence \( |u|^2 = \sum_{i=1}^n |x_iy_i|^2 \) and \( |uv|^2 = \sum_{i=1}^n |x_iy_i|^2 \). Thus:
\[ |uv|^2 \leq \sum_{i=1}^n |x_iy_i|^2 \cdot |v|^2 = |u|^2 \cdot |v|^2. \]

(c) If \( u \in I \) and \( v \in I \) we write \( v = \sum_{i=1}^n x_iy_i \), so that \( (y_i', y_j') = 0 \) for \( i \neq j \) and apply the technique of the previous paragraph.

**Lemma 8.** If \( u \in I \) then \( |u| \leq ||u||. \)

**Proof.** If \( u \in S \), then \( u = xy, x \in L, y \in R \) and \( |u| = \omega^{-2} \|x\| \cdot \|y\| \leq \omega^{-1} \|x\| \cdot \|y\| = ||u|| \) since \( \|y\| \leq \omega \|y\| \) (Lemma 2). If \( u \in I \), then \( u = \sum_{i=1}^n u_i = \sum_{i=1}^n x_iy_i \); we may assume that \( (x_i, x_j) = 0 \) for \( i \neq j \), then \( (u_i, u_j) = 0 \) and also \( |u_i, u_j| = 0 \) for \( i \neq j \) and hence \( |u|^2 = \sum_{i=1}^n |u_i|^2 \leq \sum_{i=1}^n \|u_i\|^2 = ||u||^2. \)

**Corollary.** If \( u, v \in I \) then \( [u, v] \leq ||u|| \cdot ||v||. \)

Thus the scalar product \([ , ]\) is continuous in the original topology; hence can be extended to whole \( A \). In general \( A \) is not complete in the new scalar product, so let \( \bar{A} \) be the completion of \( A \) with respect to \([ , ]\). Let us extend continuously the algebraic operations of \( A \) (including the involution) to \( \bar{A} \). Then it is easy to see that \( \bar{A} \) is an \( H^* \)-algebra.

Indeed let \( x \) be an element in \( A \) having left adjoint \( x^t \) in \( A \), then if \( z, u \in S \) we have \( z = x_1y_1, u = x_2y_2, x_i \in L, y_i \in R, i = 1, 2, \) and so
\[ [zx, u] = [x_1y_1x, x_2y_2] = \frac{1}{\omega^4} (x_1, x_2)(y_2, x_1y_1) = \frac{1}{\omega^4} (x_1, x_2)(xy_2, y_1') \]
\[ = [x_1y_1, x_2y_2x^t] = [z, ux^t]. \]

From this it is easy to verify that \([yx, z] = [y, zx^t]\) for all \( y, z \in A \). Similarly \([xy, z] = [y, x^tz]\) for all \( y, z \in A \) and it is easy to show that \( |x'| = |x| \) for all \( x \in A \) having left adjoint, from which it follows that
the involution $x \rightarrow x^t$ can be uniquely extended to whole $\bar{A}$ in such a manner that $[xy, z] = [y, x^t z]$ and $[yx, z] = [y, zx^t]$ hold for all $y, z$ in $\bar{A}$.

Now we are in a position to prove the following theorem:

**Theorem 3.** Every simple complemented algebra $A$ is isomorphic to an algebra of operators $a$ of the Hilbert Schmidt type on a Hilbert space such that $\text{tr}((aa^*)aa) < \infty$ where $\alpha$ is some (unbounded) self-adjoint operator with the domain dense in the Hilbert space.

**Proof.** Above we constructed the $\mathcal{H}$*-algebra $\bar{A}$ in which $A$ is dense. $\bar{A}$ is isomorphic to the algebra of operators of the Hilbert Schmidt type on some Hilbert space $H$ (it is easy to verify that $\bar{A}$ is simple). In particular we may take $H$ to be the closed ideal $e\bar{A}$, where $e$ is the above considered primitive left projection. The isomorphism is set up as follows: if $a \in \bar{A}$ corresponds to the operator $T$ and $x \in e\bar{A}$, then $T(x) = xa$.

Now let us consider $eA$ and $e\bar{A}$. Since the scalar product $[,]$ of $\bar{A}$ restricted to $eA$ is continuous with respect to the original norm there exists a bounded self-adjoint operator $\beta$ defined on $eA$ such that $[a, b] = (\beta(a), \beta(b))$ holds for every $a, b \in eA$. One can easily see that $\beta$ is also continuous with respect to $]|-$norm (corresponding to $[,]$): $|\beta(a)||\leq \|\beta\|\||a||$. Thus $\beta$ can be extended to whole $e\bar{A}$.

Since the mapping $a \rightarrow a^t$ is 1-1 (follows from the fact that $A$ is semi-simple), $\beta$ is 1-1 also (note that $\langle \beta(a), \beta(b) \rangle = [a, b] = \omega^{-1}(b^t, a^t)$). Since $\beta$ is also self-adjoint the range of $\beta$ (even if $\beta$ is restricted to $eA$) is dense in $eA$. Now let $x$ be any member of $eA$ and let $x_n$ be a sequence of elements in the range of $\beta$ approaching $x$ in $\|\|-$norm. Then $x_n \rightarrow x$ also in $\|\|$-norm. Let $y_n$ be the sequence such that $\beta(y_n) = x_n$. Then $|y_n - y_m| = \|\beta(y_n) - \beta(y_m)\| = \|x_n - x_m\|$, i.e. $y_n$ is a Cauchy sequence. Therefore there is an element $y$ in $e\bar{A}$ such that $y_n \rightarrow y$ in $\|\|-$norm. Then we have $x = \beta(y)$ and so the range of $\beta$ extended to $e\bar{A}$ is entire $eA$. Hence there exists an (unbounded) operator $\alpha$ with the domain dense in $e\bar{A}$ such that $[a, b] = [\alpha(a), \alpha(b)]$ holds for every $a, b \in eA$.

Let us show that $\alpha(a) = aa$ for every $a \in eA$ where $aa$ means operator defined by $\alpha(a(x))$ ($x$ is an element in the Hilbert space). But $a(x) = xa$ if $x \in e\bar{A}$. So it is sufficient to show that $\alpha(xa) = x(\alpha(a))$. But it follows from the fact that $x \in e\bar{A}$, $a \in e\bar{A}$ and $\alpha(a) \in e\bar{A}$: $\alpha(xa) = \alpha(cea) = \lambda(cea) = \lambda e\alpha(\alpha) = ex\alpha(a) = xaa$, where $\lambda$ is some scalar such that $exe = \lambda e$.

Thus we have $(a, b) = [aa, ba] = \text{tr}((ba^*)aa)$ for every $a, b \in eA$. One can quite easily show (using Lemma 1) that this is true for every $a, b \in A$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
A BOUNDARY LAYER PROBLEM FOR AN ELLIPTIC EQUATION IN THE NEIGHBORHOOD OF A SINGULAR POINT

VICTOR J. MIZEL

We consider the first boundary value problem for

\[ Lu = \epsilon \Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y) \]

on a region \( R \) under the following hypotheses

I. \( R \) is an open simply- or multiply-connected region in the \((x, y)\) plane whose boundary \( S \) consists of a finite number of simple closed curves, and \( R + S \) is contained in an open connected region \( R_0 \) throughout which \( A(x, y), B(x, y), C(x, y), \) and \( D(x, y) \) are of class \( C^6 \).

II. Along each closed curve of \( S \) the functions giving \( x, y, \) and the boundary value \( u \) in terms of arclength are of class \( C^6 \).

III. \( C(x, y) < 0 \) on \( R_0 \).

IV. The system (for characteristics of the abridged \((\epsilon = 0)\) equation)

\[
\begin{align*}
\frac{dx}{dt} &= -A(x, y), \\
\frac{dy}{dt} &= -B(x, y)
\end{align*}
\]

has as its singularities on \( R + S \) a finite number of stable attractors \( P_1, \ldots, P_n \).

Received by the editors February 14, 1956.

1 The author wishes at this point to express his gratitude to Professor N. Levinson who originally suggested the problem to him and who gave him encouragement throughout.