ON THE IMBEDDING OF A RIGHT COMPLEMENTED ALGEBRA INTO AMBROSE'S H^*-ALGEBRA

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Let A be a Banach algebra with a Hilbert space norm (norm defined by a scalar product). We shall call A a right complemented algebra if it has the property that the orthogonal complement of a right ideal is again a right ideal. This notion was introduced in the author's doctoral thesis [5]. It was proved that under certain additional assumptions every right complemented algebra is left complemented. We shall prove this theorem for a general right complemented algebra. We shall also show that the most general simple right (left) complemented algebra is of the following form.

EXAMPLE. Let α be a (possibly unbounded) self-adjoint linear operator with domain dense in a Hilbert space H and the range being a subset of H. Let A be the algebra of all linear operators a of the Hilbert Schmidt type on H such that $|a\alpha| < \infty$, where $|\ |$ is the trace norm of an operator: $|a|^2 = \mathrm{tr}\ (a^*a)$. Then A is a right (as well as left) complemented algebra in the scalar product $(a, b) = [a\alpha, b\alpha] = \mathrm{tr}\ (a\alpha(b\alpha)^*)$.

We shall use the following terminology (see [5]). A Banach algebra shall be called simple if it is semi-simple and has no proper two-sided ideals except those which are dense in whole algebra. We shall say that x^l is the left adjoint of x if $(xy, z) = (y, x^l z)$ holds for all y, z in the algebra. A left projection e is a left self-adjoint (nonzero) idempotent; a primitive left projection is a left projection which cannot be written as a sum of two doubly orthogonal left projections (compare with W. Ambrose [1]). The orthogonal complement of an ideal I will be denoted by I^p.

We have proved in [5] that every simple right complemented algebra has a primitive left projection. So we begin by proving:

THEOREM 1. *Let A be a simple right complemented algebra and let e be a primitive left projection in A. Then every element in eA has a left adjoint.*

PROOF. Let $a \in eA$; then $ea = a$. We may assume that $ae \neq 0$ (otherwise we consider $b = a + e$ for which $be \neq 0$). Then $a^2 = eaea = \lambda a$, i.e., a is a multiple of some idempotent f. Consider the closed regular right ideal $Q = \{z - fz \mid z \in A\}$, f is a relative identity of Q. We write

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$f=e_1+u$ with $e_1\in Q^p$, $u\in Q$; then e_1 is a left projection and $fe_1=f$, $e_1f=e_1$. Hence $ee_1\neq 0$ (note that $e_1e\neq 0$ also) and so we have: $ee_1=ee_1f=ee_1ef=\mu ef=\mu f$, i.e., $f^l=1/\bar{\mu}(e_1e)$. Hence a also has a left adjoint.

THEOREM 2. *The set of elements in a simple right complemented algebra A having left adjoint is dense in A.*

PROOF. Let $F=\{e_i\}$ be the family of all primitive left projections in A. Let R be the closed right ideal generalized by F, i.e., R is the closure of the linear space spanned by all elements of the form e_ix, $e_i\in F$, $x\in A$. It follows from Lemma 1 that the set of elements in R having a left adjoint is dense in R. It remains to show that $R=A$. Suppose $R\neq A$, then $R^p\neq(0)$. Let $a\in R^p$ be an element which does not have a right quasi-inverse. Consider the right regular ideal $Q=$closure of $\{ax+x\,|\,x\in A\}$ for which $-a$ is relative identity. We write $-a=e+u$ with $e\in Q^p$, $u\in Q$. Then it is easy to see that e is a left projection (of course $e\neq 0$) such that $eu=0$ (compare with [5, Lemma 2]). Thus $e\in F$ and hence $(ea, ea)=(ea, a)=0$, $ea=0$. But on the other hand $-ea=e(e+u)=e$, which is a contradiction. Thus $R=A$.

COROLLARY. *Every semi-simple right complemented algebra A is a left complemented algebra; the set of elements in A having right adjoint is dense in A.*

From now on we may refer to a semi-simple right complemented algebra simply as a "complemented algebra."

Now we proceed with the second part of our paper. Let A be a simple complemented algebra and let e be a primitive left projection in A. We consider the ideals $L=Ae$ and $R=eA$. Every element in R has a left adjoint while L has a dense subset of elements having left adjoint. We shall show that A is a dense subalgebra of a suitably constructed H^*-algebra. It will be done by proving a series of lemmas in which A, e (and hence L and R) are fixed once and for all.

LEMMA 1. *If $x_1, x_2\in L$ and $y_1, y_2\in R$, then $(x_1y_1, x_2y_2)=\omega^{-2}(x_1, x_2)\cdot(y_1, y_2)$ where $\omega=\|e\|$.*

PROOF. Since $x_2^lx_1\in eAe$ we have $x_2^lx_1=\lambda e$ for some complex λ (eAe is isomorphic to the complex field [5, Lemma 7]). Then $(x_1, x_2)=(x_1, x_2e)=(x_2^lx_1, e)=(\lambda e, e)=\lambda\|e\|^2=\lambda\omega^2$ and $(x_1y_1, x_2y_2)=(x_2^lx_1y_1, y_2)=(\lambda y_1, y_2)=\lambda(y_1, y_2)=\omega^{-2}(x_1, x_2)(y_1, y_2)$.

COROLLARY. *If $x\in L$ and $y\in R$ then $\|xy\|=\omega^{-1}\|x\|\,\|y\|$.*

LEMMA 2. *If $x\in R$ then $\|x^l\|\leqq\omega\|x\|$.*

PROOF. If $x\in R$, then $xx^l=\lambda e$ for some positive λ (we again use the

fact that eAe is isomorphic to the complex field). So we have:

$$\|x^l\|^2 = (x^l e, x^l e) = (xx^l, e) = \lambda(e, e) = \lambda\|e\|\cdot\|e\| = \|\lambda e\|\cdot\|e\|$$
$$= \|xx^l\|\cdot\|e\| \leqq \|x\|\cdot\|x^l\|\,\|e\|$$

or $\|x^l\| \leqq \|x\|\cdot\|e\| = \omega\|x\|$.

LEMMA 3. *If an element has the form* $z = \sum_{i=1}^n x_i y_i$ *with* $x_i \in L$, $y_i \in R$, *then* $x_1, x_2, \cdots, x_n$ *can be so chosen in* L *that* $(x_i, x_j) = 0$ *for* $i \neq j$; *also* $y_1, y_2, \cdots, y_n$ *can be so chosen in* R *that* $(y_i^l, y_j^l) = 0$ *for* $i \neq j$.

PROOF. The lemma is easily proved by induction.

Now consider $S = LR = AeA$. We define the function $[\ ,\ ]$ on $S \times S$ by setting

$$[x_1x_1, y_2y_2] = \frac{1}{\omega^4}(x_1, x_2)(y_2^l, y_1^l) \qquad \text{where } \omega = \|e\|.$$

(It is understood that $x_1, x_2 \in L$, $y_1, y_2 \in R$.)

LEMMA 4. *The function* $[\ ,\ ]$ *is independent of the choice of the primitive left projection* e.

PROOF. Let e_1 and e_2 be any two primitive left projections. Suppose $z_i = x_i y_i$, $i = 1, 2$, with $x_i \in Ae_1$ and $y_i \in e_1 A$. Then $[x_1y_1, x_2y_2]_1 = 1/\omega_1^4(x_1, x_2)(y_2^l, y_1^l)$, where $\omega_1 = \|e_1\|$. We shall show that $z_i \in Ae_2A$ and that $[z_1, z_2]_1 = [z_1, z_2]_2$, where $[\ ,\ ]_2$ is the above function defined with respect to e_2.

It can be easily shown that there are elements e_{12} and e_{21} in A such that $e_{12}^l = e_{21}$, $e_{12}e_{21} = e_1$, $e_{21}e_{12} = e_2$, $e_{12}e_2e_{21} = e_1$ and $e_{21}e_1e_{12} = e_2$. Then $z_i = x_iy_i = x_ie_1y_i = x_ie_{12}e_2e_{21}y_i$ and hence $z_i \in Ae_2A$. Also

$$[z_1, z_2]_2 = \frac{1}{\|e_2\|^4}(x_1e_{12}, x_2e_{12})(y_2^le_{12}, y_1^le_{12})$$
$$= \frac{1}{(e_{21}e_{12}, e_{21}e_{12})^2}\cdot\frac{(x_1, x_2)(e_{12}, e_{12})}{\omega_1^2}\cdot\frac{(y_2^l, y_1^l)(e_{12}, e_{12})}{\omega_1^2}$$
$$= \frac{1}{(e_{12}, e_{12})^2}\cdot\frac{1}{\omega_1^4}(x_1, x_2)(y_2^l, y_1^l)(e_{12}, e_{12})^2$$
$$= \frac{1}{\omega_1^4}(x_1, x_2)(y_2^l, y_1^l) = [z_1, z_2]_1.$$

LEMMA 5. *The function* $[\ ,\ ]$ *has the following properties*:

(a) $[\lambda x, y]=\lambda[x, y]$
(b) $[x, y]=$ *complex conjugate of* $[y, x]$.
(c) $[x, x]\geqq 0$ *and* $[x, x]=0$ *if and only if* $x=0$.
(d) $[\sum_{i=1}^n z_i, z]=\sum_{i=1}^n [z_i, z]$, *provided* $z_i, z\in S$ *and* $\sum_{i=1}^n z_i\in S$.

PROOF. (a)–(c) are easily verified. We shall prove (d). Since $z_i, z\in S$ we have $z_i=x_iy_i$, $z=xy$ and also $u=\sum_{i=1}^n z_i=vw$ with $x_i, x, v\in L$, $y_i, y, w\in R$. Let us assume that $z_1, z_2, \cdots, z_n, x$ are fixed while y is variable. We have: $(u, z)=(vw, xy)=\omega^{-2}(v, x)(w, y)$ or $(v, x)(w, y)=\omega^2\sum_{i=1}^n (x_iy_i, xy)=\sum_{i=1}^n (x_i, x)(y_i, y)$. Now let us assume that $(v, x)\neq 0$. This can be done without loss of generality. Then we can write $(x_i, x)=\lambda_i(v, x)$ for some complex λ_i, $i=1, 2, \cdots, n$ and so we have:

$$(v, x)(w, y) = \sum_{i=1}^n \lambda_i(v, x)(y_i, y) = (v, x)\sum_{i=1}^n (\lambda_i y_i, y)$$

or $(w, y)=(\sum_{i=1}^n \lambda_i y_i, y)$. It can be written $(w-\sum_{i=1}^n \lambda_i y_i, y)=0$, where y is an arbitrary element in R. This simply means that $w=\sum_{i=1}^n \lambda_i y_i$ (note that $w, y\in R$).

Now let us take y so that $z=xy$. Then we have:

$$[u, z] = [vw, xy] = \frac{1}{\omega^4}(v, x)(y^l, w^l) = \frac{1}{\omega^4}(v, x)\left(y^l, \sum_{i=1}^n \bar{\lambda}_i y_i^l\right)$$

$$= \frac{1}{\omega^4}\sum_{i=1}^n \lambda_i(v, x)(y^l, y_i^l) = \frac{1}{\omega^4}\sum_{i=1}^n (x_i, x)(y^l, y_i^l)$$

$$= \sum_{i=1}^n [x_i, y_i xy] = \sum_{i=1}^n [z_i, z].$$

Now let I be the set of all finite sums of elements in S, i.e., I is the set of all elements of the form $\sum_{i=1}^n x_iy_i$ with $x_i\in L$, $y_i\in R$. It is easy to see that I is a two-sided ideal dense in A.

LEMMA 6. *The function* $[\ ,\ ]$ *has a unique extension to* I, *which has the properties of a scalar product.*

PROOF. If $z=\sum_{i=1}^n z_i$ and $u=\sum_{j=1}^n u_j$ with $z_i\in S$, $u_j\in S$, then we define $[z, u]=\sum_{i,j} [u_i, z_j]$. It is easy to verify that $[\ ,\ ]$ is a scalar product, using Lemma 5. The uniqueness of $[\ ,\ ]$ follows from (d).

LEMMA 7. *If* $u, v\in I$ *then* $|uv|\leqq|u||v|$, *where* $|\ \ |$ *denotes the corresponding to* $[\ ,\ ]$ *norm.*

PROOF. (a) We first take $u, v\in S$, then $u=x_1y_1$, $v=x_2y_2$ with $x_i\in L$, $y_i\in R$. Then $uv=x_1y_1x_2y_2=\lambda x_1ey_2$, since $y_1x_2=\lambda e$ for some λ, and

$$|uv| = |\lambda|\,|x_1 y_2| = \frac{1}{\omega^2}|\lambda|\cdot\|x_1\|\cdot\|y_2^l\|.$$

But $|\lambda|\omega^2 = |\lambda|(e, e) = |(\lambda e, e)| = |(y_1x_2, e)| = |(x_2, y_1^l)| \leqq \|x_2\|\,\|y_1^l\|$, i.e., $|\lambda| \leqq \omega^{-2}\|x_2\|\cdot\|y_1^l\|$.

Hence

$$|uv| \leqq \frac{1}{\omega^2}\|x_1\|\cdot\|y_1^l\|\,\frac{1}{\omega^2}\|x_2\|\cdot\|y_2^l\| = |u|\,|v|.$$

(b) Suppose that $u\in I$ and $v\in S$; then $u=\sum_{i=1}^n x_iy_i$. We may assume that $(x_i, x_j)=0$ for $i\neq j$. Then $[x_iy_i, x_jy_j]=0$ and $[x_iy_iv, x_jy_jv]=0$ and hence $|u|^2=\sum_{i=1}^n |x_iy_i|^2$ and $|uv|^2=\sum_{i=1}^n |x_iy_iv|^2$. Thus:

$$|uv|^2 \leqq \sum_{i=1}^n |x_iy_i|^2|v|^2 = |u|^2|v|^2.$$

(c) If $u\in I$ and $v\in I$ we write $v=\sum_{i=1}^n x_iy_i$ so that $(y_i^l, y_j^l)=0$ for $i\neq j$ and apply the technique of the previous paragraph.

LEMMA 8. *If $u\in I$ then $|u|\leqq\|u\|$.*

PROOF. If $u\in S$, then $u=xy$, $x\in L$, $y\in R$ and $|u|=\omega^{-2}\|x\|\cdot\|y^l\| \leqq \omega^{-1}\|x\|\cdot\|y\|=\|u\|$ since $\|y^l\|\leqq\omega\|y\|$ (Lemma 2). If $u\in I$, then $u=\sum_{i=1}^n u_i=\sum_{i=1}^n x_iy_i$; we may assume that $(x_i, x_j)=0$ for $i\neq j$, then $(u_i, u_j)=0$ and also $[u_i, u_j]=0$ for $i\neq j$ and hence $|u|^2 = \sum_{i=1}^n|u_i|^2\leqq\sum_{i=1}^n\|u_i\|^2=\|u\|^2$.

COROLLARY. *If $u, v\in I$ then $[u, v]\leqq\|u\|\cdot\|v\|$.*

Thus the scalar product $[\ ,\ ]$ is continuous in the original topology; hence can be extended to whole A. In general A is not complete in the new scalar product, so let $\tilde{A}$ be the completion of A with respect to $[\ ,\ ]$. Let us extend continuously the algebraic operations of A (including the involution) to $\tilde{A}$. Then it is easy to see that $\tilde{A}$ is an H^*-algebra.

Indeed let x be an element in A having left adjoint x^l in A, then if $z, u\in S$ we have $z=x_1y_1$, $u=x_2y_2$, $x_i\in L$, $y_i\in R$, $i=1, 2$, and so

$$[zx, u] = [x_1y_1x, x_2y_2] = \frac{1}{\omega^4}(x_1, x_2)(y_2^l, x^ly_1^l) = \frac{1}{\omega^4}(x_1, x_2)(xy_2^l, y_1^l)$$

$$= [x_1y_1, x_2y_2x^l] = [z, ux^l].$$

From this it is easy to verify that $[yx, z]=[y, zx^l]$ for all $y, z\in A$. Similarly $[xy, z]=[y, x^lz]$ for all $y, z\in A$ and it is easy to show that $|x^l|=|x|$ for all $x\in A$ having left adjoint, from which it follows that

the involution $x \to x^l$ can be uniquely extended to whole $\tilde{A}$ in such a manner that $[xy, z] = [y, x^l z]$ and $[yx, z] = [y, zx^l]$ hold for all y, z in $\tilde{A}$.

Now we are in a position to prove the following theorem:

THEOREM 3. *Every simple complemented algebra A is isomorphic to an algebra of operators a of the Hilbert Schmidt type on a Hilbert space such that* $\mathrm{tr}((a\alpha)^* a\alpha) < \infty$ *where α is some (unbounded) self-adjoint operator with the domain dense in the Hilbert space.*

PROOF. Above we constructed the H^*-algebra $\tilde{A}$ in which A is dense. $\tilde{A}$ is isomorphic to the algebra of operators of the Hilbert Schmidt type on some Hilbert space H (it is easy to verify that $\tilde{A}$ is simple). In particular we may take H to be the closed ideal $e\tilde{A}$, where e is the above considered primitive left projection. The isomorphism is set up as follows: if $a \in \tilde{A}$ corresponds to the operator T and $x \in e\tilde{A}$, then $T(x) = xa$.

Now let us consider eA and $e\tilde{A}$. Since the scalar product $[\ ,\ ]$ of $\tilde{A}$ restricted to eA is continuous with respect to the original norm there exists a bounded self-adjoint operator β defined on eA such that $[a, b] = (\beta(a), \beta(b))$ holds for every $a, b \in eA$. One can easily see that β is also continuous with respect to $|\ \ |$-norm (corresponding to $[\ ,\ ]$): $|\beta(a)| = \|\beta^2(a)\| \leqq \|\beta\|\ \|\beta(a)\| = \|\beta\|\ |a|$. Thus β can be extended to whole $e\tilde{A}$.

Snce the mapping $a \to a^l$ is 1-1 (follows from the fact that A is semi-simple), β is 1-1 also (note that $(\beta(a), \beta(b)) = [a, b] = \omega^{-4}(b^l, a^l)$). Since β is also self-adjoint the range of β (even if β is restricted to eA) is dense in eA. Now let x be any member of eA and let x_n be a sequence of elements in the range of β approaching x in $\|\ \ \|$-norm. Then $x_n \to x$ also in $|\ \ |$-norm. Let y_n be the sequence such that $\beta(y_n) = x_n$. Then $|y_n - y_m| = \|\beta(y_n) - \beta(y_m)\| = \|x_n - x_m\|$, i.e. y_n is a Cauchy sequence. Therefore there is an element y in $e\tilde{A}$ such that $y_n \to y$ in $|\ \ |$-norm. Then we have $x = \beta(y)$ and so the range of β extended to $e\tilde{A}$ is entire eA. Hence there exists an (unbounded) operator α with the domain dense in $e\tilde{A}$ such that $(a, b) = [\alpha(a), \alpha(b)]$ holds for every $a, b \in eA$.

Let us show that $\alpha(a) = a\alpha$ for every $a \in eA$ where $a\alpha$ means operator defined by $\alpha(a(x))$ (x is an element in the Hilbert space). But $a(x) = xa$ if $x \in e\tilde{A}$. So it is sufficient to show that $\alpha(xa) = x(\alpha(a))$. But it follows from the fact that $x \in e\tilde{A}$, $a \in e\tilde{A}$ and $\alpha(a) \in e\tilde{A}$: $\alpha(xa) = \alpha(exea) = \alpha(\lambda ea) = \lambda\alpha(ea) = \lambda e\alpha(ea) = exe\alpha(a) = x\alpha(a)$, where λ is some scalar such that $exe = \lambda e$.

Thus we have $(a, b) = [a\alpha, b\alpha] = \mathrm{tr}\ ((b\alpha)^* a\alpha)$ for every $a, b \in eA$. One can quite easily show (using Lemma 1) that this is true for every $a, b \in A$.

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A BOUNDARY LAYER PROBLEM FOR AN ELLIPTIC EQUATION IN THE NEIGHBORHOOD OF A SINGULAR POINT[1]

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We consider the first boundary value problem for

$$Lu = \epsilon\Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y)$$

on a region R under the following hypotheses

I. R is an open simply- or multiply-connected region in the (x, y) plane whose boundary S consists of a finite number of simple closed curves, and $R+S$ is contained in an open connected region R_0 throughout which $A(x, y)$, $B(x, y)$, $C(x, y)$, and $D(x, y)$ are of class C^6.

II. Along each closed curve of S the functions giving x, y, and the boundary value $\bar{u}$ in terms of arclength are of class C^6.

III. $C(x, y) < 0$ on R_0.

IV. The system (for characteristics of the abridged ($\epsilon = 0$) equation)

$$\frac{dx}{dt} = -A(x, y), \qquad \frac{dy}{dt} = -B(x, y) \tag{1}$$

has as its singularities on $R+S$ a finite number of stable attractors $P_1, \cdots, P_n$.

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