SOME CONSEQUENCES OF THE APPROXIMATION THEOREM OF BING

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1. **Introduction.** In the following paragraphs we consider several unsolved problems in the topology of 3-space. Among the results is included a new formulation of properties characterizing tame curves in three-space without any explicit mention of polyhedral sets.

Three-space is denoted by R^3, the null-set by $\square$, and the set of points in A but not in B by $A\backslash B$. The combinatorial boundary of chain E is ∂E and the set of points on such a boundary by $|\partial E|$. The closure of a set A is denoted by Cl A, or $\overline{A}$.

THEOREM (Bing). *Given a compact 2-manifold M of R^3 with or without boundaries, a compact subset D of R^3 and a positive number ϵ. There is a 2-manifold N and a homeomorphism f of M on N such that no point of M is moved more than ϵ and N is locally polyhedral at each point of $N\backslash D$. If $D\neq\square$, f may also be taken so that no point x is moved more than the minimum of ϵ and the distance from x to D. In particular, f is the identity map at all points of $D\cap M$ and $D\cap M=D\cap N$* [1].

2. **The "orthogonal" disk problem.** Some simple closed curves in R^3 have the property that, if J is the curve, there is a disk D whose boundary links J and such that J and D meet in a single point. This property is preserved by homeomorphisms acting on R^3. The class of curves having this property is known to include curves $\{J\}$ such that under some homeomorphism of R^3 on itself some sub-arc of J is mapped on a rectifiable arc [5]. By the use of his approximation theorem, R. H. Bing [2] has constructed an example of a curve that admits no orthogonal disk.

3. **A new formulation of properties $\mathcal{P}$ and $\mathcal{Q}$.** A 1-manifold J is said to have property $\mathcal{P}$ if at each point x of J and for positive ϵ there is a set $K(x, \epsilon)$ of diameter less than ϵ that is a topological 2-sphere whose interior contains x, that meets J at a set of points whose cardinal does not exceed the order of x in J, and such that K is locally polyhedral modulo J.

To see that the locally polyhedral character of K may be omitted

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suppose K has the other requirements and $\epsilon' < \epsilon -$diameter K. Using $M = K$, $D = J$ and ϵ' in the approximation theorem, then $N = K_1(x, \epsilon)$ has the original requirements.

A 1-manifold J is said to have property $\mathfrak{Q}$ at x if there is a disk D_1 such that $J \cap D_1$ is the closure of a neighborhood of x in J and D_1 is locally polyhedral at the points of $D_1 \backslash J$. To see that the locally polyhedral character of D_1 may be omitted, suppose D_2 has the requirements of D_1 apart from locally polyhedral character. If $M = D_2$ and $D = J$ and ϵ arbitrary in the approximation theorem we find a disk $N = D_1$ that is locally polyhedral at points of $N \backslash J$ and has the other requirements.

4. **1-cells contained in the interior of 2-cells.** It has been known since 1921 when Antoine's thesis appeared that an arc in R^3 may be a subset of no topological 2-cell in R^3. Such an arc certainly does not lie on the boundary of a 2-cell and still less on the boundary of a 3-cell. (That a 2-cell in R^3 may be a subset of no topological 2-sphere and hence lie on the boundary of no 3-cell is implied by a result announced by Kapuano [**7**]). We limit ourselves here chiefly to the special case where the 1-cell lies interior (relative) to a 2-cell.

A fairly simple proof is now available that arcs exist lying on no 2-cell. Example 1.4 of Fox-Artin [**3**] is an example of a wild arc that is the union of two tame arcs, having only an end point in common. If J is the set of points on this arc and J_1 is a sub-arc of J whose interior contains the "wild point" of the imbedded arc, then J_1 is also wild. Now if J were a part of the boundary of some disk D in R^3, then J_1 has properties $\mathfrak{P}$ and $\mathfrak{Q}$ (in the new form) and, by Theorem 6 of [**6**], would be tame. But J_1 is wild.

THEOREM. *Let J be a closed 1-cell in R^3, then J lies in the interior of a closed 2-cell if and only if J lies on the boundary of some bounded, open, connected ulc^1 subset of R^3.*

If: Let A be the bounded, open, connected subset of R^3 that is ulc^1 and has J on its boundary. By the results of Wilder, [**8**, Theorem 8.3, p. 311], every nondegenerate component of the boundary M of A is a classical 2-manifold. If M_1 is the component of M determined by J, then, by triangulation of M_1, an open 2-cell E in M_1 may be found such that J is in Int E (rel M_1).

Only if: If J lies in the interior of a disk E, there is a sub-disk E_1 of E such that J is in $|\partial E_1|$. Using the approximation theorem (with $D = J$, $M = E_1$) there is an "almost" polyhedral disk N whose boundary contains J that is locally polyhedral at points of $N \backslash J$. One may "inflate" N to obtain a closed 3-cell F that is locally polyhedral at

points $F\backslash J$ and $|\partial F|$ contains J. Then J is contained in the closure of the interior of F and Int F is the required bounded, open, connected, ulc^1 subset.

5. **A special case of the union of two locally tame sets.** It is known that locally tame sets are tame. However, the union of two tame sets having a single common point need not be tame. In this connection the following theorem is of interest.

THEOREM. *Let J be a simple closed curve that is polyhedral and let J be the boundary of any disk F in R^3. Then J is the boundary of a polyhedral disk.*

(Thus J is tame and unknotted).

PROOF. If J is the boundary of any disk D, then J has the (new) property $\mathcal{Q}$ at each point. Since J is polyhedral, it has property $\mathcal{P}$ at each point. By Theorem VII of [6] it is locally tamely imbedded, hence tamely imbedded. By Graeub [**4**, p. 39], this implies there is a polyhedral 2-cell of which J is the boundary.

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