ON INFINITELY DIFFERENTIABLE POSITIVE DEFINITE FUNCTIONS

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1. Suppose that \( f(x) \) is an infinitely differentiable positive definite function. That is to say

\[
f(x) = \int_{-\infty}^{\infty} e^{itz} d\alpha(t),
\]

where \( d\alpha(t) \) is a bounded non-negative measure. Since \( f(x) \) is infinitely differentiable, it is well known (cf. C.-G. Esseen [4, p. 24]) that

\[
f^{(n)}(x) = \int_{-\infty}^{\infty} i^n e^{itz} d\alpha(t).
\]

Therefore, the sequence \( \{(−i)^n f^{(n)}(0)\} \) represents a Hamburger moment sequence. Again, if \( \{\xi_k\} \) is an arbitrary finite set of complex numbers and \( m \) is any non-negative integer, then

\[
\left| \sum_{k=0}^{n} \xi_k (-i)^k f^{(k+m)}(x) \right|^2 = \left| \int_{-\infty}^{\infty} t^m e^{itz} \sum_{k=0}^{n} \xi_k t^k d\alpha(t) \right|^2 
\]

\[
\leq \int_{-\infty}^{\infty} t^{2m} d\alpha(t) \int_{-\infty}^{\infty} \left| \sum_{k=0}^{n} \xi_k t^k \right|^2 d\alpha(t) 
\]

\[
= M_m \sum_{r=0}^{n} \sum_{s=0}^{n} \xi_r \xi_s (-i)^{r+s} f^{(r+s)}(0),
\]

where \( M_m = (−i)^{2m} f^{(2m)}(0) \).

It turns out that if we add to these two necessary conditions a third condition, namely that \( \{(−i)^n f^{(n)}(0)\} \) is a determined Hamburger moment sequence,\(^2\) then these three conditions are sufficient for an infinitely differentiable function to have the representation (1). However, even more is true. If \( f(x) \) is defined and infinitely differentiable on some open interval containing the origin and satisfies the above conditions, then it has the representation (1); i.e. it can be extended to be a positive definite function. Since the Hamburger moment sequence is determined the extension is clearly unique (cf.}

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\(^2\) By this we mean that there exists a unique non-negative measure \( d\alpha(t) \) such that \((−i)^{nf^{(n)}(0)} = f^{(n)}(0) = f^{(n)}(0) = \int_{-\infty}^{\infty} t^n d\alpha(t)\).

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Esseen [4, pp. 24–25]). Moreover, we shall show that if the Hamburger sequence is not determined, the first two necessary conditions are not sufficient. On the other hand the fact that the Hamburger sequence \( \{(-i)^n f^{(n)}(0)\} \) is determined is in general not a necessary condition.

The theorem we shall prove in this note is as follows:

**Theorem.** Let \( f(x) \) be an infinitely differentiable function defined on the open interval \((-a, b)\) where \( a, b > 0 \). If

(a) \( \{(-i)^k f^{(k)}(0)\}_{k=0}^{n} \) is a determined Hamburger moment sequence and
(b) for every non-negative integer \( m \) there exists an \( M_m > 0 \) such that for every \( x \in (-a, b) \) and every finite set \( \{\xi_k\}_{k=0}^{n} \) of complex numbers

\[
\left| \sum_{k=0}^{n} \xi_k (-i)^k f^{(k+m)}(x) \right| \leq M_m \sum_{r=0}^{n} \sum_{s=0}^{n} \xi_r \xi_s (-i)^{r+s} f^{(r+s)}(0),
\]

then there exists a bounded non-negative measure \( d\alpha(t) \) such that

\[
f(x) = \int_{-\infty}^{\infty} e^{itx} d\alpha(t).
\]

As a tool in the proof of this theorem we shall use the theory of operators in Hilbert space. This theorem was inspired by a recent result of A. P. Calderon and A. Devinatz [2; 3] when we noticed, that after some preliminary work, the same methods as used in [2] and [3] could be used to obtain our more general result.

2. In this section we shall construct the requisite tool which we shall use in the proof of our theorem, namely a Hilbert space. To do this we consider the class \( \mathcal{F}' \) of functions of the form

\[
g(x) = \sum_{k=0}^{n} \xi_k (-i)^k f^{(k)}(x).
\]

If \( h(x) \) is another element of \( \mathcal{F}' \), namely

\[
h(x) = \sum_{k=0}^{m} \eta_k (-i)^k f^{(k)}(x),
\]

we shall construct an inner product in \( \mathcal{F}' \) by the formula

\[
(g, h) = \sum_{r=0}^{n} \sum_{s=0}^{m} \xi_r \eta_s (-i)^{r+s} f^{(r+s)}(0).
\]

To show that this is a well defined function, suppose that \( g \) and \( h \) have different representations; i.e.
\[ g(x) = \sum_{k=0}^{n} \xi_k (-i)^k f^{(k)}(x), \quad h(x) = \sum_{k=0}^{m} \eta_k (-i)^k f^{(k)}(x). \]

Then
\[ \sum_{r=0}^{n'} \sum_{r'=0}^{m'} \xi_{r'} \eta_r (-i)^{r+s} f^{(r+s)}(0) = \sum_{r=0}^{n'} \xi_r (-i)^{r+s} f^{(r+s)}(0) = \sum_{s=0}^{m} \eta_s (-i)^s g^{(s)}(0) \]
\[ = \sum_{r=0}^{m} \sum_{s=0}^{m} \xi_{r} \eta_{s} (-i)^{r+s} f^{(r+s)}(0). \]

This shows that the bilinear function in (2) is well defined. Moreover, since \( \{(-i)^n f^{(n)}(0)\} \) is a moment sequence, \( (g, g) \geq 0 \) and by condition (b) of the theorem if \( (g, g) = 0 \), then \( g(x) = 0 \). Conversely if \( g(x) = 0 \), \( (g, g) = 0 \). Therefore, the bilinear function defined in (2) is an actual inner product on \( \mathcal{F}' \).

In general, \( \mathcal{F}' \) is not complete with respect to this norm. We shall show that it can be completed to a Hilbert space \( \mathcal{F} \) of functions on \( (-a, b) \). Suppose then that \( \{g_n\} \subset \mathcal{F}' \) is a Cauchy sequence in \( \mathcal{F}' \). That is to say \( \|g_n - g_m\| \to 0 \) as \( n, m \to \infty \). By condition (b) of the theorem \( |g_n(x) - g_m(x)| \) goes uniformly to zero as \( n, m \to \infty \). Therefore, there exists a continuous function \( g(x) \) defined on \( (-a, b) \) such that \( |g_n(x) - g(x)| \to 0 \) as \( n \to 0 \). This extended class of functions, which we get as pointwise limits of Cauchy sequences in \( \mathcal{F}' \), we shall designate by \( \mathcal{F} \). It is clear that \( \mathcal{F} \) is a linear space over the complex number field. It remains to extend the inner product from \( \mathcal{F}' \) to \( \mathcal{F} \) so that \( \mathcal{F} \) becomes a Hilbert space. If \( g, h \in \mathcal{F} \), there exist Cauchy sequences \( \{g_n\}, \{h_n\} \subset \mathcal{F}' \) such that \( g_n(x) \to g(x) \) and \( h_n(x) \to h(x) \). We shall define

\[ (g, h) = \lim_{n \to \infty} (g_n, h_n). \]

That this limit exists is clear since

\[ |(g_n, h_n) - (g_m, h_m)| \leq \|g_n - g_m\| \|h_n\| + \|g_m\| \|h_n - h_m\|. \]

The quantities \( \|h_n\| \) and \( \|g_m\| \) are uniformly bounded and therefore \( \{(g_n, h_n)\} \) is a Cauchy sequence. We must show yet that (3) is a well defined function.

Suppose that \( \{\bar{g}_n\}, \{\bar{h}_n\} \subset \mathcal{F}' \) are Cauchy sequences such that \( \bar{g}_n(x) \to g(x) \) and \( \bar{h}_n(x) \to h(x) \) uniformly in \( (-a, b) \). Condition (b) of the theorem tells us that for any \( m, g^{(m)}_n(x) \to g^{(m)}(x) \) and \( h^{(m)}_n(x) \to h^{(m)}(x) \) uniformly in \( (-a, b) \) and therefore, in particular, \( g^{(m)}_n(0) \)
→g^{(m)}(0) and \( \hat{h}^{(m)}_n(0) \rightarrow h^{(m)}(0) \). Now, clearly

\[
\lim_{n \to 0} (g_n, h_n) = \lim_{n \to \infty} \lim_{m \to \infty} (g_n, h_m) = \lim_{m \to \infty} \lim_{n \to \infty} (g_n, h_m).
\]

Further,

\[
\lim_{n, m \to \infty} (g_n, h_m) - \lim_{n, m \to \infty} (\bar{g}_n, \bar{h}_m) = \lim_{n \to \infty} \lim_{m \to \infty} (g_n - \bar{g}_n, h_m) + \lim_{m \to \infty} \lim_{n \to \infty} (\bar{g}_n, h_m - \bar{h}_m).
\]

Suppose that

\[
h_m(x) = \sum_k \xi_{k, m}(-i)^k f^{(k)}(x)
\]

and

\[
\bar{g}_n(x) = \sum_k \eta_{k, n}(-i)^k f^{(k)}(x).
\]

Then we have

\[
\lim_{m \to \infty} \lim_{n \to \infty} (g_n - \bar{g}_n, h_m) = \lim_{m \to \infty} \lim_{n \to \infty} \sum_k \xi_{k, m}(-i)^k [g^{(k)}_n(0) - \bar{g}^{(k)}_n(0)] = 0
\]

and similarly

\[
\lim_{n \to \infty} \lim_{m \to \infty} (\bar{g}_n, h_m - \bar{h}_m) = \lim_{n \to \infty} \lim_{m \to \infty} \sum_k \eta_{k, n}(i)^k [h^{(k)}_m(0) - \bar{h}^{(k)}_m(0)] = 0.
\]

This shows that the function defined in (3) is indeed well defined.

To show that this bilinear function is an inner product we first note that for every \( g \) in \( \mathcal{F} \) there exists a Cauchy sequence \( \{g_n\} \subset \mathcal{F}' \) such that

\[
(g, g) = \lim_{n \to \infty} (g_n, g_n) \geq 0.
\]

Again, suppose that for \( g \in \mathcal{F} \), \( (g, g) = 0 \). Suppose \( \{g_n\} \subset \mathcal{F}' \) is a Cauchy sequence such that \( g_n(x) \rightarrow g(x) \) and \( (g, g) = \lim (g_n, g_n) = 0 \). Then since

\[
|g_n(x)|^2 = \left| \sum_k \xi_{k, n}(-i)^k f^{(k)}(x) \right|^2 \leq M_0 ||g_n||^2
\]

we have that \( g_n(x) \rightarrow 0 \), which shows that \( g(x) \equiv 0 \). On the other hand if \( g(x) \equiv 0 \) on \((-a, b)\) then \( g \in \mathcal{F}' \) and \( (g, g) = 0 \). Therefore, \( \mathcal{F} \) forms a linear space with an inner product. The proof of the fact that \( \mathcal{F} \) is complete uses standard arguments and we leave this to the reader.

An important fact that we shall need in the future, a fact used
previously in connection with the space $\mathcal{F}'$, is that for any $g \in \mathcal{F}$

$$(-i)^n g^{(n)}(0) = (g(x), (-i)^n f^{(n)}(x)).$$

This can be easily proved by taking a Cauchy sequence $\{g_n(x)\} \subseteq \mathcal{F}'$ such that $g_n(x) \to g(x)$, noting that (4) is true for every $g_n$ in this sequence and then passing to the limit.

3. Now that we have constructed the Hilbert space $\mathcal{F}$ we can proceed with the proof of our theorem. First we wish to set up a conjugation operator on $\mathcal{F}$. Consider first an element $g \in \mathcal{F}'$; i.e. $g(x) = \sum_0^\infty \xi_k(-i)^k f^{(k)}(x)$. Define

$$Jg(x) = \sum_0^n \xi_k(-i)^k f^{(k)}(x).$$

Now, $Jg(x) = g(x)$ and if $h(x) = \sum_0^m \eta_k(-i)^k f^{(k)}(x)$, then

$$(Jg, Jh) = \sum_{r=0}^n \sum_{s=0}^m \xi_r\eta_s (-i)^{r+s} f^{(r+s)}(0) = (h, g).$$

Since $\mathcal{F}'$ is dense in $\mathcal{F}$, $J$ can be extended to all of $\mathcal{F}$ and is a conjugation operator.

Define an operator $D$, with domain $\mathcal{F}'$, by the relation

$$Dg(x) = -idg(x)/dx.$$ 

In other words, if $g(x) = \sum_0^n \xi_k(-i)^k f^{(k)}(x)$, then

$$Dg(x) = \sum_0^n \xi_k(-i)^{k+1} f^{(k+1)}(x).$$

If $h(x) = \sum_0^m \eta_k(-i)^k f^{(k)}(x)$, then

$$(Dg, h) = \sum_{r=0}^n \sum_{s=0}^m \xi_r\eta_s (-i)^{r+s+1} f^{(r+s+1)}(0) = (g, Dh).$$

Therefore, $D$ is a symmetric operator. Further, since it clearly permutes with $J$, it has a self-adjoint extension. We shall show that the self-adjoint extension is unique and is therefore the closure of $D$.

Suppose $H$ is any self-adjoint extension of $D$, and $dE(t)$ its canonical resolution of the identity. If $f_0(x) = f(x)$, then

$$(-i)^n f^{(n)}(0) = (H^n f_0, f_0) = \int_{-\infty}^{\infty} t^n dE(t) f_0, f_0).$$

Since by hypothesis $\{-i)^n f^{(n)}(0)\}$ is a uniquely determined Hamburger moment sequence, the measure $d(E(t)f_0, f_0)$ is uniquely deter-
minded. Suppose we let \( f_n(x) = (-i)^n f^{(n)}(x) \). The linear manifold generated by this class of elements is dense in \( \mathcal{F} \). Now,

\[
(E(\lambda)f_n, f_m) = (E(\lambda)H^{n+m}f_0, f_0) = \int_{-\infty}^{\lambda} t^{n+m}d(E(t)f_0, f_0).
\]

Therefore, for any \( n \) and \( m \), \( (E(\lambda)f_n, f_m) \) is uniquely determined in the sense that if \( H_1 \) is another self-adjoint extension of \( D \), \( dE_1(t) \) its canonical spectral measure, then \( (E_1(\lambda)f_n, f_m) = (E(\lambda)f_n, f_m) \). This means however that \( D \) has only one self-adjoint extension, namely its closure.

What we have just proved means that \( D^* \), the adjoint of \( D \), is self-adjoint and is the closure of \( D \). Therefore \( g \) is in the domain of \( D^* \) if and only if there exists a sequence \( \{g_n\} \subset \mathcal{F}' \) such that \( g_* \to g \) and \( Dg_n \to D^*g \) in the strong topology of \( \mathcal{F} \). This implies uniform pointwise convergence and therefore,

\[
D^*g(x) = -idg(x)/dx.
\]

Let us consider the group of unitary operators

\[
U_x = \int_{-\infty}^{\infty} e^{itz}dE(t),
\]

where \( dE(t) \) is the canonical spectral measure of \( D^* \). Let \( g \in \mathcal{F} \) be such that

\[
U_xg = \int_{-c}^{c} e^{itz}dE(t)g,
\]

where \( c \) is a positive finite number. It is clear that any such element belongs to the domain of \( D^* \). Let us expand \( e^{itz} \), as a function of \( t \), in its Taylor series about the origin. Since this Taylor series is uniformly convergent in any finite interval we have for every \( x \)

\[
U_xg = \sum_{0}^{\infty} \frac{x^n}{n!} \int_{-c}^{c} i^{n}f^{n}dE(t)g
\]

\[
= \sum_{0}^{\infty} \frac{x^n}{n!} i^{n}D^*g_n,
\]

where the convergence is in the strong topology of \( \mathcal{F} \). But since convergence in the strong topology implies pointwise convergence we have for every \( y \in (-a, b) \)

\[
U_xg(y) = \sum_{0}^{\infty} \frac{g^{(n)}(y)}{n!} x^n.
\]
Since this series has an infinite radius of convergence for every \( y \in (-a, b) \) it is well known (cf. R. P. Boas [1]) that \( g(y) \) is analytic. Therefore if \( x+y \in (-a, b) \)
\[ U_xg(y) = g(y + x) . \]  
Since the class of elements for which (5) holds is dense in \( \mathcal{F} \), (5) must hold for every element of \( \mathcal{F} \). If again we set \( f_0(x) = f(x) \) then by means of the relationship (4) we get
\[ f(x) = U_xf(0) = (U_xf_0, f_0) = \int_{-\infty}^{\infty} e^{itz}d(E(t)f_0, f_0) . \]
This completes the proof of the theorem.

4. In this section we shall show that there exist functions \( f(x) \) such that \( \{(-i)^n f^{(n)}(0)\} \) is an undetermined Hamburger moment sequence and which satisfy condition (b) of our theorem but which are not positive definite. This is essentially the same example as given in [3].

Let \( \{\mu_n\}_{0}^{\infty} \) be any undetermined Hamburger moment sequence. Then there exist two different bounded non-negative measures \( d\alpha_1(t) \) and \( d\alpha_2(t) \) such that
\[ \mu_n = \int_{-\infty}^{\infty} t^n d\alpha_1(t) = \int_{-\infty}^{\infty} t^n d\alpha_2(t) . \]
Let
\[ f_1(x) = \int_{-\infty}^{\infty} e^{itz} d\alpha_1(t) , \]
\[ f_2(x) = \int_{-\infty}^{\infty} e^{itz} d\alpha_2(t) . \]
Further, let
\[ f(x) = \begin{cases} f_1(x) & \text{for } x \geq 0 , \\ f_2(x) & \text{for } x \leq 0 . \end{cases} \]
Then, \( (-i)^n f^{(n)}(0) = \mu_n \) and condition (b) of the theorem is clearly satisfied. However, \( f(x) \) cannot be a positive definite function since a positive definite function must satisfy the relation
\[ f(x) = \bar{f}(-x) . \]
However, we would get in this case
\[ f_1(x) = \bar{f}_1(-x) = \bar{f}_2(-x) = f_2(x) \]
which would mean \( d\alpha_1 = d\alpha_2 . \)
ON A SERIES OF RAINVILLE INVOLVING
LEGENDRE POLYNOMIALS

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1. The object of this paper is to obtain some relations involving
Legendre polynomials with the help of a series given by E. D. Rain-
ville. The results are believed to be new.

2. We start with the series given by E. D. Rainville

\[ P_n(\cos \alpha) = \frac{(\sin \alpha)}{\sin \beta} \sum_{k=0}^{n} c_{n,k} \left( \frac{\sin (\beta - \alpha)}{\sin \alpha} \right)^{n-k} P_k(\cos \beta). \]

Putting \( \beta = 2\alpha \) and \( \cos 2\alpha = x \), we get

\[ 2^{n/2}(1 + x)^{n/2}P_n \left( \left( \frac{1 + x}{2} \right)^{1/2} \right) = \sum_{k=0}^{n} c_{n,k}P_k(x). \]

From (2.2) and the orthogonal property

\[ \int_{-1}^{1} (1 + x)^{n/2}P_r(x)P_n \left( \left( \frac{1 + x}{2} \right)^{1/2} \right) dx \]

\[ = \frac{c_{n,\gamma}}{2^{n/2-1}(2\gamma + 1)}, \quad 0 \leq \gamma \leq n, \]

\[ = 0, \quad r > n. \]

Using (2.3) with Adams’ expansion (Modern analysis, p. 331) for

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