II. Is it possible (for certain $n$) to replace the number $\lfloor n/4 \rfloor + 1$ of the theorem by a number $M(n)$ which is greater than $\lfloor n/4 \rfloor + 1$?

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ON THE MULTIPLICATIVE GROUP OF A DIVISION RING

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Let $K$ be a noncommutative division ring with center $Z$ and multiplicative group $K^*$. Hua [2; 3] proved that (i) $K^*/Z^*$ is a group without center, and (ii) $K^*$ is not solvable. A generalization (Theorem 1) will be given here which contains as a special case (Theorem 2) the fact that $K^*/Z^*$ has no Abelian normal subgroups. This latter theorem obviously contains both (i) and (ii). As a further corollary it is shown that if $M$ and $N$ are normal subgroups of $K^*$ not contained in $Z^*$, then $M \cap N$ is not contained in $Z^*$. The final theorem is that an element $x$ outside $Z$ contains as many conjugates as there are elements in $K$. This makes more precise a theorem of Herstein [1], who showed that $x$ has an infinite number of conjugates.

Square brackets will denote multiplicative commutation. If $S$ is a set, then $o(S)$ will mean the number of elements in $S$. A subgroup $H$ of $K^*$ is subinvariant in $K^*$ if there is a chain $\{N_i\}$ of subgroups such that $H \lhd N_{r-1} \lhd \cdots \lhd N_1 \lhd K^*$, where $A \lhd B$ means that $A$ is a normal subgroup of $B$.

Lemma. Let $K$ be a division ring, $H$ a nilpotent subinvariant subgroup of $K^*$, $y \in H$, $x \in K^*$, and $[y, x] = \lambda \in Z^*$, $\lambda \neq 1$. Then the field $Z(x)$ is finite.

Proof. The proof of this lemma is essentially part of Hua’s proof of (ii), but will be included for the sake of completeness.

Let $f$ be any rational function over $Z$ such that $f(x) \neq 0$. Then
\[ x_1 = [y, f(x)] = yf(x)y^{-1}f(x)^{-1} = f(yxy^{-1})f(x)^{-1} = f(\lambda x)f(x)^{-1}; \]
\[ x_2 = [y, x_1] = f(\lambda x)f(\lambda x)^{-2}f(x); \]
and, by induction, if \( x_n = [y, x_{n-1}] \), then
\[ x_n = \prod_{i=0}^{n} f(\lambda^i x)^{(-1)^{n-i}(\mathbb{Q})}. \]

Now, by the subinvariance of \( H \), \( x_1 \in N_1, x_2 \in N_2, \ldots, x_r \in H \), and since \( H \) is nilpotent, \( x_n = 1 \) for some \( n \). Letting \( f(x) = 1 + x \), we have
\[ \prod (1 + \lambda^i x)^{(\mathbb{Q})} - \prod (1 + \lambda^i x)^{(\mathbb{Q})} = 0, \]
where the first product is taken over those \( i \) such that \( n-i \) is even and the second over those \( i \) such that \( n-i \) is odd, \( i = 0, \ldots, n \). In the left member, the constant term is equal to 0, while the coefficient of \( x \) is
\[ \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \lambda^i = (\lambda - 1)^n \neq 0. \]
Thus \( x \) is algebraic (of degree at most \( 2^{n-1} - 1 \)) over \( Z \). If \( c \in Z^* \), then \([y, cx] = \lambda \), and \( cx \) is also a root of (1). Therefore \( Z \) is finite, and \( Z(x) \) is finite as asserted.

**Theorem 1.** Let \( K \) be a division ring, \( G \) and \( H \) subinvariant subgroups of \( K^* \), \( x \in G \), \( y \in H \), and \([y, x] = \lambda \in Z^* \lambda \neq 1 \). Then one of \( G \) and \( H \) is not nilpotent.

**Proof.** Deny the theorem. By the lemma \( Z \) is finite and both \( x \) and \( y \) are algebraic over \( Z \). Since \( yx = \lambda xy \), the set \( S \) of elements of the form \( \sum z_{ij}x^iy^j \), \( z_{ij} \in Z \), is a finite noncommutative division ring.

**Theorem 2.** If \( K \) is a noncommutative division ring, then \( K^*/Z^* \) has no normal Abelian subgroups.

**Proof.** If \( N/Z^* \) is a normal Abelian subgroup, then \( N \) is a nilpotent normal subgroup of \( K^* \). The division ring generated by \( N \) is invariant, hence by the Cartan-Brauer-Hua theorem is \( K \) itself. Therefore \( N \) is non-Abelian, and there are elements \( x, y \in N \) such that \([y, x] = \lambda \in Z^* \lambda \neq 1 \). This contradicts Theorem 1.

**Remark.** The proof of Theorem 2 depends on Wedderburn’s theorem that a finite division ring is a field. This can be avoided by the following considerations. Using the notation of the preceding proofs, \( x_1 = [y, 1 + x] \in N, x_2 = [y, x_1] \in Z, x_3 = [y, x_2] = 1 \), and \( n \leq 3 \). However, the coefficient of \( x^4 \) in the left member of (1) vanishes, so that \( x \) and \( y \) are of degree 2 over \( Z \), and \( o(Z) = 3 \) since it contains the distinct
elements 0, 1, and \( \lambda \). Then \( o(S) \leq 3^4 \), and since \( S \) must have room for a center and a subfield not in the center, \( o(S) = 3^4 \) and \( Z \) is the center of \( S \). Thus \( S^* \) is a group of order 80 and contains an element \( u \) of order 5. The centralizer \( C(u) \) of \( u \) in \( S \) is a division ring, hence of order 3, 9, or 81, therefore by Lagrange's theorem of order 81. But then \( u \in Z, o(Z) \geq 5 \), and the contradiction proves the theorem.

**Theorem 3.** Let \( K \) be a division ring and \( M \) and \( N \) be normal subgroups of \( K^* \) not contained in \( Z^* \). Then \( M \cap N \) is not contained in \( Z^* \).

**Proof.** Deny the theorem. Then \([M, N] \subset Z^* \). Let \( y \in N, y \in Z^* \).
Since the centralizer \( C(M) \) of \( M \) is an invariant division ring not \( K \), by the Cartan-Brauer-Hua theorem, \( C(M) = Z \). Hence \( y \in C(M) \), and there is an \( x \) in \( M \) such that \([y, x] = \lambda \neq 1, \lambda \in Z^* \). The map \( a \sigma = [x, a] \) is a homomorphism of \( N \) into \( Z^* \) with kernel \( L \subseteq [N, N] \). Since \( y \in L, y \in [N, N] \), and since \( y \) was arbitrary, \([N, N] \subset Z^* \). Therefore \( N \) is nilpotent. Similarly \( M \) is nilpotent and Theorem 1 is contradicted.

**Lemma.** Let \( K \) be an infinite division ring (perhaps commutative), \( D \) a proper subdivision ring. Then \([K^*:D^*] = o(K) \).

**Proof.** Let \( x \in K^*, x \in D^* \). Then the cosets \( D^*(x+a), a \in D \), are distinct. Hence \([K^*:D^*] \supseteq o(D) \). If \( o(D) = o(K) \) we are done; if not, then \( o(K) = o(K^*) = o(D^*)[K^*:D^*] \), hence again \([K^*:D^*] = o(K) \).

**Theorem 4** (See [1]). If \( K \) is a division ring and \( x \) is a noncentral element, then \( x \) has \( o(K) \) conjugates.

**Proof.** The centralizer \( C \) of \( x \) is a proper subdivision ring of \( K \). Then the number of conjugates of \( x \) equals \([K^*:C^*] \) which is \( o(K) \) by the lemma.

**Bibliography**


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