MODULES WITHOUT INVARIANT BASIS NUMBER

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In a recent paper,¹ [p. 190] the author constructed a ring over which a finitely based module has invariant basis number if and only if it has a basis of length <2. It was indicated that this construction was generalizable, and this proceeds as follows:

Let \( R \) be a word ring (with unit) in symbols \( \{a_{ij}, b_{st}\} \) \((j, s = 1, \ldots, n \geq 2; i, t = 1, \ldots, n + 1)\) over the rationals. Let \( A \) and \( B \) be matrices whose elements are respectively \( \{a_{ij}\} \) and \( \{b_{st}\} \), and consider the relations of

\[
AB - I_{n+1} = 0, \quad BA - I_n = 0.
\]

It is clear that the method of proof [Lemma 1, p. 191] is applicable, and thus for each \( \alpha \in R \) we may obtain in a finite number of steps a unique normal form \( N(\alpha) \) not containing any of the leading words of the left-hand members of relations (1). If \( H \) is the two-sided ideal whose basis is the set of all elements of \( AB - I_{n+1} \) and \( BA - I_n \), we accordingly have an effective means of deciding whether or not \( \alpha \in H \) (namely \( \alpha \in H \) if and only if \( N(\alpha) = 0 \)). The quotient ring \( K = R/H \) may also be regarded as a word ring in \( \{a_{ij}, b_{st}\} \) all of whose members are reduced to normal form.

It may also be verified that \( K \) contains no zero divisors, the proof following that of [Lemma 2, p. 192]. Note that if the degree \( d[\alpha] \) is defined to be the length of the longest word in \( \alpha \), this proof also shows that \( d[\alpha \beta] = d[\alpha] + d[\beta] \). According to the remarks of [p. 193, footnote] it is clear that a module over \( K \) with a basis of length 1 has invariant basis number, while a module with a basis of length \( \geq n \) does not. This leaves the question open for a module over \( K \) with basis of length \( q \) \( (1 < q < n) \). It is the purpose of the present paper to show that such a module also has invariant basis number.

A word \( y \) is said to be similar to \( x \) if it differs from \( x \) only in either or both (a) the first subscript of its first symbol, (b) the second subscript of its last symbol; \( y \) is left \( \{ \) right \( \} \) similar to \( x \) if only (a) \( \{ \) (b) \( \} \) applies. Let \( \{m_i\} \) \((i = 1, \ldots, s)\) be a partition of the integer \( m \). If \( x \) is a word of length \( m \), a word \( y \) is said to be compatible with \( x \) (relative to the partition \( \{m_i\} \) if \( x = x_1 x_2 \cdots x_s \) and \( y = y_1 y_2 \cdots y_s \), where \( d[x_i] = d[y_i] = m_i \) with \( y_1 \) right similar to \( x_1 \), \( y_s \) left similar to \( x_s \), and

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¹ William G. Leavitt, Modules over word rings, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 188–193. References to this paper will be enclosed in brackets.
y_i similar to x_i \ (1 < i < s). Note that if u and v are words (in normal form) such that \( d[u] = \sum_i m_i \) and \( d[v] = \sum_{i+1} m_i \), then if the product uv is compatible with x before normalization, all its longest words remain compatible after normalization. An \( \alpha \in K \) is homogeneous if its words are all of the same length.

Suppose \( \{\alpha_i, \beta_i\} (i = 1, \ldots, q; 1 < q < n) \) is a set of homogeneous members of \( K \) (in normal form) such that \( d[\alpha_1] \geq d[\alpha_2] \geq \cdots \geq d[\alpha_q] \) and such that for all \( i \) we have \( d[\alpha_i\beta_i] = m \). Suppose further that all longest words in any of the products \( \alpha_i\beta_i \) are compatible relative to the partition of \( m \) given by the nonzero numbers in the sequence

\[
(2) \quad d[\alpha_2], d[\alpha_2-1] - d[\alpha_2], \ldots, d[\alpha_1] - d[\alpha_2], d[1].
\]

**Lemma 1.** If the above set \( \{\alpha_i, \beta_i\} \) also satisfies the condition \( d[\sum \alpha_i\beta_i] < m \), then (possibly relabeling those \( \alpha_i \) for which \( d[\alpha_i] = d[\alpha_i] \)) there exist \( \{\theta_i\} (i = 2, \ldots, q) \) such that \( d[\alpha_1 + \sum \alpha_i\theta_i] < d[\alpha_1] \).

If any \( \alpha_k \) is constant, the lemma follows trivially, for we may take \( \theta_k = -\alpha_k/\alpha_k, \theta_i = 0 \) (\( i \neq k \)). Similarly, if \( \beta_i \) is constant we may choose \( \theta_i = \beta_i/\beta_i \). Accordingly we suppose that all \( d[\alpha_i], d[\beta_i] > 0 \).

By compatibility the words of any particular \( \alpha_i \) all end with either \( a_{pi} \) or \( b_{pi} \) (for fixed \( p \)), those of \( \beta_i \) all begin with either \( a_{q_i} \) or \( b_{q_i} \) (for fixed \( q \)). Call a product \( \alpha_i\beta_i \), of type \( ab \) if the words of \( \alpha_i \) end in \( a_{pi} \) while those of \( \beta_i \) begin with \( b_{q_i} \). A similar description applies to products of type \( ba, aa, \) or \( bb \).

Let us suppose that the partition of \( m \) mentioned above has \( s \) members, then also by compatibility the words of \( \alpha_1 \) have in general \( 2s - 3 \) variable indices, while those of \( \beta_i \) have a single variable index. Call these indices \( i, \ldots, j \) and \( k \). If \( d[\alpha_1] = \cdots = d[\alpha_i] > d[\alpha_{i+1}] \geq \cdots \), this also applies to all \( \alpha_i, \beta_i \) for which \( 1 \leq r \leq t \). Let the coefficient of the word (of such an \( \alpha_r \)) with variable indices \( i, \ldots, j \) be \( c_i^* \ldots c_j^* \), and of the word of \( \beta_i \), with index \( k \) be \( d_i^{*k} \). If \( \alpha_i\beta_i \), is a product of type \( ab \) or \( ba \), the coefficients of its longest words are \( c_i^* \ldots d_j^{*k} - \delta_j k c_i^* \ldots d_j^{*k} \) (where \( \delta_j k = 1 \) if \( j = k \), 0 otherwise), while if the product is of type \( aa \) or \( bb \) the second term is omitted. Proceeding to those \( \alpha_r \) for which \( d[\alpha_r] = d[\alpha_{r+1}] < d[\alpha_i] \), the words of such an \( \alpha_r \) have \( 2s - 5 \) variable indices \( i, \ldots \) and those of the corresponding \( \beta_r \) have three \( \cdot \cdot \cdot jk \). If such an \( \alpha_i\beta_i \) is a product of type \( ab \) or \( ba \) its longest words have coefficients \( c_i^* \ldots d_j^{*k} - \delta_j k c_i^* \ldots d_j^{*k} \) where the subscripts of \( \delta \), are the last of \( c_i^* \) and the first of \( d_j^{*k} \). Again, if the product is of type \( aa \) or \( bb \), the second term is omitted. Similar expressions are evidently obtained for the coefficients of the longest words of all \( \alpha_i\beta_i \) (\( i = 1, \ldots, q \)).

Now \( q < n \) and each variable index has range either \( 1 \) to \( n \) or \( 1 \) to
so each variable index has at least \( q+1 \) values. Also \( d[\alpha_q] > 0 \), so at least one \( \alpha_q \neq 0 \) (say, for definiteness, it is among \( \alpha_2, \ldots, \alpha_{q+1} \)).

We now choose, for the moment, the subscripts \( i, \ldots, j \) as follows: the second and succeeding subscripts will be fixed, with the second equal to 1, the last \( q+1 \), and no two adjacent subscripts equal. Since

\[
d[\alpha_1 + \cdots + \alpha_{q+1}] < m,
\]

it follows that all the coefficients of its longest words are zero. Thus, with the above choice of subscripts, we have

\[
(3) \ c_i \cdots d_{q+1} + \cdots + c_i d_{1} \cdots q+1k = 0,
\]

where we choose \( i=2, 3, \ldots, q+1 \) and \( k=1, \ldots, q \).

Since at least one of the \( \alpha_i \neq 0 \), the rank of the coefficient matrix of (3) is at least one. Thus the set of vectors \( (d_1, \ldots, d_{q+1}) \) is dependent. We distinguish the following cases:

Case I. \( d_1 = \cdots = d_q = 0 \). Since \( d[\beta_1] > 0 \), at least one, say \( d_q \neq 0 \).

Then for any choice of the subscripts \( i, \ldots, j \) we have

\[
(4) \ c_i \cdots d_u + \cdots + c_i \cdots d_u + c_i \cdots d_j + \cdots - \delta_i c_{i+1}^{q+1} \delta_i c_{i+1}^{q+1} = 0.
\]

We choose \( \theta_r = d_u / d_q \) for \( r \leq t \), while for \( r > t \) we form the words of \( \theta_r \) by dropping from the words of \( \beta_1 \) the portions similar to \( \beta_1 \). If the variable subscripts of a resulting word are \( \cdots j \), we choose for its coefficient \( d_r \cdots j \) (that is, the coefficient of \( c_r \cdots \) in the equation derived from (4) by dividing by \( d_q \)).

Case II. Some \( d_u \neq 0 \) (by relabelling, this becomes \( d_1 \neq 0 \)) and for some \( u \) all \( d_1 = \cdots = d_u = 0 \). Then for any choice of subscripts \( i, \ldots, j \) we have

\[
(5) \ c_i \cdots d_1 + \cdots + c_i \cdots d_1 + \cdots + c_i d_1 \cdots q+1 = 0
\]

\[
-j \neq 1,
\]

Again \( \theta_r = d_u / d_q \) \( (r \leq t) \), and for \( r > t \) the coefficients of \( \theta_r \) (with subscripts \( \cdots j \)) are \( d_r \cdots j / d_1 \) if \( j \neq 1 \) and \( -d_r \cdots u / d_1 \) if \( j = 1 \).

Case III. For each \( k \) at least one of \( d_k, \cdots, d_q \neq 0 \). Since the vectors \( (d_k, \cdots, d_q) \) are dependent, there must be some \( u \) such that \( (d_u, \cdots) = \sum_{k=1}^{u-1} h_k (d_k, \cdots) \) for some set of constants \( \{ h_k \} \).

Remark that we are considering the worst situation, namely that in which all of the products \( \alpha_i \beta_1 \) are of type \( ab \) or \( ba \). It is clear that a similar, though simpler, treatment may be used when some of the products are of type \( aa \) or \( bb \). This is especially so if \( \alpha_i \beta_1 \) is of this type, for then we have essentially only the following Case I.
By relabelling, let $d_i^t \neq 0$. Then

$$
(6) \quad c_i - j \theta_k + \cdots + c_i \theta_k - \delta_{i \neq j} \theta_k = 0 
$$

where $\theta_k = \theta_k^d / \theta_k^1 (r \leq t)$ and the coefficients of the words of $\theta_r (r > t)$ are $\theta_r - \theta_t / \theta_t^1 (j \neq t)$.

If the equations of (7) are multiplied respectively by $h_k$ and summed, the result is

$$
(8) \quad c_i \theta_t + \cdots + c_i \theta_t - \delta_{i \neq t} \theta_t = 0
$$

where $\epsilon_r - \epsilon_t = \sum_{k=1}^{u-1} h_k \theta_t^r - \theta_t^1 (j \neq t)$.

It is clear from (4), (5), or (6) and (8) (depending on which of the three cases applies), the above choice of $\theta_r$ gives coefficients of the longest words of $\sum \phi_r \theta_r$ which are just sufficient to cancel the coefficients $c_i e_t$ of the words of $\phi_i$.

We now define an elementary transformation to be either a permutation (relabelling) $\{ \phi_i, \theta_i \} \rightarrow \{ \phi_i, \theta_i \}$ or a transformation of type $\{ \phi_i, \theta_i \} \rightarrow \{ \phi_i', \theta_i' \}$ where for some $r$

$$
(9) \quad \begin{align*}
\phi_i' &= \phi_i + \sum_{r+1}^{q} \phi_i \theta_k, \\
\theta_i' &= \theta_i \\
\phi_i' &= \phi_i (i \neq r), \\
\theta_i' &= \theta_i - \theta_i \theta_r (i > r).
\end{align*}
$$

**Lemma 2.** Let $\{ \phi_i, \theta_i \}$ ($i = 1, \cdots, q < n$) be members of $K$ arranged so that $d[\phi_i] \geq d[\phi_{i+1}]$. If $m = \max d[\phi_i \theta_i]$ and $\sum \phi_i \theta_i < m$, then there exist $\{ \phi_i^*, \theta_i^* \}$, reached by a finite sequence of elementary transformations, such that $\sum \phi_i^* \theta_i^* = \sum \phi_i \theta_i$ and $\max d[\phi_i^* \theta_i^*] < m$.

Since the case $q = 1$ is impossible, it is sufficient to show that from case $q$ follows either a case $\leq q - 1$ or the lemma. If for any $i$ we have $d[\phi_i \theta_i] < m$, then we already have a case $\leq q - 1$. Thus suppose all $d[\phi_i \theta_i] = m$. We again use the partition (2) determined by the $d[\phi_i]$ and $d[\theta_i]$. Let $x$ be a longest word of $\phi_i$ and $y$ of $\theta_i$, and let $\alpha_i$ be the part of $\phi_i$ containing all (and only) words compatible with $x$ relative to this partition. Similarly, $\beta_i$ is the part of $\theta_i$ compatible with $y$. In general, $\alpha_i$ is the part of $\phi_i$ compatible with $x$ or a first portion of $x$, while $\beta_i$ is the part of $\theta_i$ compatible with $uy$, where $u$ is either 1 or a last portion of $x$. Clearly, all longest words of $\alpha_i \beta_i$ are compatible with $xy$, and are not combinable with any other word.
obtained from any product $\alpha_i\beta_i$. Thus $d[\sum \alpha_i\beta_i] < m$, while $d[\alpha_i\beta_i] = m$ for all $i$, and so the set $\{\alpha_i, \beta_i\}$ satisfies the conditions of Lemma 1. Hence (possibly after a permutation) there exist $\{\beta_i\}$ such that $d[\alpha_1 + \sum_i \alpha_i\beta_i] < d[\alpha_1]$. Now let

$$\begin{align*}
\alpha'_1 &= \alpha_1 + \sum_i \alpha_i\beta_i, \\
\beta'_1 &= \beta_1,
\end{align*}$$

then $\sum \alpha'_i\beta'_i = \sum \alpha_i\beta_i$, and all words compatible with $x$ have been eliminated from $\alpha_1$.

This means that $\alpha_i\beta_1$ (dropping primes henceforth, for convenience) now contains no word compatible with $xz$ for any word $z$ (with $d[z] = d[y]$). Thus if now $\sum \alpha_i\beta_i$ contains such a word, equating its coefficient to zero will involve nothing from $\alpha_i\beta_i$. We consider first the case $d[\alpha_2] = d[\alpha_1]$. If $\alpha_2$ contains a word compatible with $x$, either $d[\alpha_2\beta_2] < m$, in which case the lemma is proved, or all words compatible with $x$ may be eliminated from (possibly a permuted) $\alpha_2$ in exactly the manner above. This process may clearly be extended to eliminate $x$ from all words $\alpha_i$ $(i = 1, \ldots, t)$ for which $d[\alpha_i] = d[\alpha_1]$.

Now suppose $d[\alpha_{i+1}] < d[\alpha_1]$ and $x = uv$, where $d[u] = d[\alpha_{i+1}]$. If $\alpha_{i+1}$ contains a word compatible with $u$, while $\beta_{i+1}$ contains a word compatible with $vz$ (for any $z$), equating to zero coefficients of words in $\sum \alpha_i\beta_i$ compatible with $xz$ again involves nothing from $\alpha_i\beta_i$ $(i \leq t)$. Thus, in a similar way, $u$ may be eliminated from (a possibly permuted) $\alpha_{i+1}$. Thus we have (or could obtain) a situation in which either $\alpha_{i+1}$ contains no word compatible with $u$, or $\beta_{i+1}$ contains no word compatible with $vz$. Clearly this process may be continued until for each $i$ either $\alpha_i$ contains no word compatible with $u_i$, or $\beta_i$ contains no word compatible with $vz$ (where $x = u\alpha_i$ and $d[u_i] = d[\alpha_i]$). Thus all words compatible with $xz$ have been eliminated from all $\alpha_i\beta_i$.

We now proceed by induction. We suppose that for a given $r$ for which $d[\alpha_{r+1}] < d[\alpha_r]$, for any word $u$ such that $d[u] = d[\alpha_r]$ we are able to perform a series of elementary transformations such that no word of $\alpha_1, \alpha_2, \ldots, \alpha_r$ begins with a word compatible with $u$, and if $u = u_i\alpha_i$ with $d[u_i] = d[\alpha_i]$ $(i > r)$ for each $i$, either $\alpha_i$ contains no word compatible with $u_i$, or no word of $\beta_i$ begins with a word compatible with $v_i$.

Now suppose $d[hv] = d[\alpha_r]$ and $d[h] = d[\alpha_{r+1}]$, and we carry out the elimination described in the induction hypothesis. If $hw$ is another such word, we wish to show that such an elimination relative to $hw$ does not destroy that already accomplished relative to $hv$. The trans-
formations are of the type $\alpha_i \rightarrow \alpha_i + \sum_{j+1}^{g} \alpha_i \theta_i$ and $\beta_i \rightarrow \beta_i - \theta_i \beta_j$ ($i > j$), for some fixed $j$. If $j \leq r$ we know that no $\alpha_i$ ($i \leq r$) begins with a word compatible with $hv$, so could not restore such a word to $\alpha_i$. Also for $i > r$ either (I) $\alpha_i$ contains no word compatible with the first part of $hv$ or (II) $\beta_i$ contains no word whose first part is compatible with the final part of $hv$. But in forming the words of $\theta_i$ we take the words of $\bar{\beta}_i$ and lop off the part compatible with $\bar{\beta}_j$. Since the remaining part is at least as long as the final part of $hv$, it follows that if (II) applies to $\beta_i$ then it also applies to $\theta_i$. Thus in either case $\alpha_i \theta_i$ cannot restore a word whose first portion is compatible with $hv$. Also, for $i > r$, if $\beta_i$ contains no word whose first part is compatible with the last of $hv$, this is clearly also true of $\beta_i - \theta_i \beta_j$.

Now when $j > r$, we are engaged in removing from $\alpha_j$ words compatible with $h$ or a first portion of $h$. Thus no such word would be restored to $\alpha_j$. Furthermore, for any $j > r$ for which such elimination is to be performed, the corresponding $\beta_j$ cannot contain a word whose first portion is compatible with the end of $hv$. Thus if $\beta_i$ has no such word, $\beta_i - \theta_i \beta_j$ cannot.

We can thus eliminate as described above, all words beginning with a word compatible with $hw$, for any $w$, from all $\alpha_i$ ($i \leq r$); that is, all words beginning with a word compatible with $h$. Also if $\alpha_{r+1}$ has a word compatible with $h$, then $\beta_{r+1}$ has no longest word beginning with any $w$. This would mean $d[\alpha_{r+1}, \beta_{r+1}] < m$ and the lemma would follow. Thus suppose $\alpha_{r+1}$ has no word compatible with $h$, and similarly for the remaining $\alpha_i$ for which $d[\alpha_i] = d[\alpha_{r+1}]$. Finally, if $h = h_i w_i$ with $d[h_i] = d[\alpha_i]$ (and $d[w_i] \neq 0$), either $\alpha_i$ has no word compatible with $h_i$ or $\beta_i$ has no word whose first part is compatible with $w_i w$ for any $w$. This establishes the induction.

By induction, then, for any $g$ for which $d[g] = d[\alpha_g]$, we may eliminate all words beginning with a word compatible with $g$ from $\alpha_i$ ($i = 1, \cdots, q$). Clearly no further elimination can restore such a word to any $\alpha_i$ and hence we may eventually eliminate all longest words from some $\alpha_i$. Then $d[\alpha_i \beta_i] < m$ and the lemma follows.

**Lemma 3.** If $\sum_i \alpha_i \beta_i = 1$ then there exists a finite sequence of elementary transformations to $\{\alpha_i^+, \beta_i^+\}$ such that $\alpha_i^+$ is a constant and $\alpha_i^+ = 0$ ($i \geq 2$).

By a permutation $\sum \alpha_i \beta_i$ may be placed in condition to apply Lemma 2, so there exists $\sum \alpha_i \beta_i = 1$ such that $\max d[\alpha_i \beta_i] < \max d[\alpha_i \beta_i]$ (dropping * for convenience). But suppose $\max d[\alpha_i \beta_i] > 0$. Since $\sum \alpha_i \beta_i = 1$, at least one $\alpha_i \beta_i = \text{constant}$. By permutation we get $\alpha_n = k$
and we can eliminate all other $\alpha_i$. Another permutation gives $\alpha_i = k, \alpha_i = 0$ (all $i \geq 2$).

**Theorem.** For each $n > 1$ there exists a ring without zero divisors over which a finitely based module has invariant basis number if and only if it has a basis of length $< n$.

According to previous remarks we need consider only case $n > 2$ and we need only prove that a module with basis of length $q < n$ has invariant basis number. This will be true if for any $m$ by $q$ and $q$ by $m$ ($m > q$) matrices $P$ and $Q$, the relation $PQ = I_m$ is impossible.

Suppose $PQ = I_m$. The first row of $P$ and first column of $Q$ satisfies Lemma 3. Clearly a nonsingular matrix $T$ exists such that the first row of $P' = PT$ and first column $Q' = T^{-1}Q$ receive any desired elementary transformation. Since the first row of $P'$ is $(k_1, 0 \cdots 0)$ and $P'Q' = I_m$, it follows that the first row of $Q'$ is $(1/k_1, 0 \cdots 0)$. By a similar process, applied to the second row and second column of $P'$ and $Q'$, we reach $P'', Q''$ whose second rows are all zero except for the first two elements. But by this process we would reach $P*Q* = I_m$, where the last $m - q$ columns of $Q*$ are zero.

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