IDEALS IN A CERTAIN BANACH ALGEBRA

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1. Introduction. The purpose of this paper is to improve a result of Bertram Yood in his paper [2]. We are concerned with a complex commutative Banach algebra $X$ and a compact Hausdorff space $\Omega$. Let $C(\Omega, X)$ denote the set of all continuous functions defined over $\Omega$ with values in $X$. $C(\Omega, X)$ is a complex commutative $B$-algebra if we define addition, multiplication, and scalar multiplication in the natural “pointwise” manner and if $\|f\|_{C(\Omega, X)} = \sup_{a \in \Omega} |f(a)|_X$ for $f \in C(\Omega, X)$. Here, $\| \cdot \|_X$ denotes the norm in $X$. Yood proves, in Lemma 5.1 of his paper, that if $X$ has a unit and if every maximal ideal in $C(\Omega, X)$ is of the form $\{f \in C(\Omega, X) \mid f(a_0) \in M_0\}$ with $a_0 \in \Omega$ and $M_0$ a maximal ideal in $X$, then $\mathfrak{M}(C(\Omega, X))$, the space of maximal ideals in $C(\Omega, X)$ topologized in the Gelfand sense, is homeomorphic with $\Omega \times \mathfrak{M}(X)$, i.e., the topological product of $\Omega$ with the space of maximal ideals in $X$. We will show that all the maximal ideals in $C(\Omega, X)$ are necessarily of the above form and, further, Yood’s result on $\mathfrak{M}(C(\Omega, X))$ is true even if $X$ lacks a unit. If $X$ lacks a unit element, then $C(\Omega, X)$ lacks a unit and $\mathfrak{M}(X)$, $\mathfrak{M}(C(\Omega, X))$ denote the spaces of all regular maximal ideals in $X$ and $C(\Omega, X)$ respectively.

2. Proof of theorem. Before establishing two lemmas which we will employ in proving our theorem, we point out that the $B$-algebra $X$ can be considered as contained as a subset of $C(\Omega, X)$. This is accomplished merely by identifying each $x \in X$ with the constant map $\Omega \rightarrow x$. The elements of $X$ will thus be viewed as constants in $C(\Omega, X)$.

Lemma 1. Let $f \in C(\Omega, X)$ and $\varepsilon > 0$. Then there exist $x_i \in X$ and continuous complex-valued functions $f_i$ defined over $\Omega$ ($i = 1, 2, \ldots, n$) such that $\|f - \sum_{i=1}^{n} x_i f_i\|_{C(\Omega, X)} \leq \varepsilon$.

Proof. Let $S = f(\Omega)$. $S$ is a compact set in $X$ since $\Omega$ is compact and $f$ is continuous. Take spheres of radius $\varepsilon$ about each point in $S$. This is an open covering of $S$ and, since $S$ is compact, there is a finite open
subcovering \( S(x_i, \epsilon) \) of \( S (i = 1, 2, \cdots, n) \), \( x_i \in S \). We may find a partition of unity subordinate to this covering, i.e., we may find continuous functions on \( S, \lambda_i(x) \) \( (i = 1, 2, \cdots, n) \), such that \( 0 \leq \lambda_i(x) \leq 1 \), \( \lambda_i(x) = 0 \) on \( S - S(x_i, \epsilon) \) and \( \sum_{i=1}^{n} \lambda_i(x) \equiv 1 \) for \( x \in S \). To see this, define \( \mu_i(x) = \text{dist}(x, S - S(x_i, \epsilon)) \) for \( i = 1, 2, \cdots, n \) where \( \text{dist}(x, S - S(x_i, \epsilon)) \) denotes the distance from \( x \) to the set \( S - S(x_i, \epsilon) \). Each \( \mu_i(x) \) is continuous and \( \geq 0 \). We see that \( \mu_i(x) = 0 \) if \( x \in S - S(x_i, \epsilon) \), \( \mu_i(x) > 0 \) if \( x \in S(x_i, \epsilon) \). Write \( \mu(x) = \sum_{i=1}^{n} \mu_i(x) \). Clearly \( \mu(x) \neq 0 \) for all \( x \in S \) since \( \{S(x_i, \epsilon)\} \) is a covering of \( S \). Define \( \lambda_i(x) = \mu_i(x)/\mu(x) \) for \( x \in S \). These functions make up the desired partition of unity.

Let \( C(x_1, x_2, \cdots, x_n) \) denote the convex hull of the \( x_i \)'s, i.e., \( y \in C(x_1, x_2, \cdots, x_n) \) if and only if there exist numbers \( r_1, r_2, \cdots, r_n \geq 0 \) such that \( \sum_{i=1}^{n} r_i = 1 \) and \( y = \sum_{i=1}^{n} r_i x_i \). Using the \( \lambda_i(x) \) we may define \( T: S \rightarrow C(x_1, x_2, \cdots, x_n) \) as follows: \( Tx = \sum_{i=1}^{n} \lambda_i(x) x_i \) \( (x \in S) \). \( T \) is continuous and, further, \( |Tx - x| \leq \epsilon \) for all \( x \in S \). For, \( |Tx - x| = |\sum_{i=1}^{n} \lambda_i(x) x_i - x| = |\sum_{i=1}^{n} \lambda_i(x) x_i - (\sum_{i=1}^{n} \lambda_i(x)) x| = |\sum_{i=1}^{n} \lambda_i(x) (x_i - x)| \leq \sum_{i=1}^{n} \lambda_i(x) |x_i - x| \leq \epsilon \). To complete the proof, define \( f_i = \lambda_i(f) \). Then, \( |Tf(a) - f(a)| = |\sum_{i=1}^{n} x_i \lambda_i(f(a)) - f(a)| \leq \epsilon \) for all \( a \in \Omega \). q.e.d.

**Lemma 2.** Every, not identically zero, continuous multiplicative linear functional in the \( \mathcal{B} \)-algebra \( C(\Omega, X) \) is of the form \( \phi(f) = \phi_M(f(a)) \) for some \( a \in \Omega, M \in \mathfrak{M}(X) \). Here, \( \phi_M \) denotes the Gelfand homomorphism from \( X \) onto the complex numbers associated with \( M \).

**Proof.** Suppose, firstly, that \( X \) has a unit \( e \) and that \( |e| = 1 \) (if \( |e| \) were not 1 we could renorm \( X \) so as to achieve this). Then \( \phi \) is a nonzero continuous multiplicative linear functional on \( eC(\Omega) = \{ef \in C(\Omega, X) | f \in C(\Omega) \} \) (here, \( C(\Omega) \) denotes the \( \mathcal{B} \)-algebra of continuous complex-valued functions defined over \( \Omega \)). Also \( \phi \) is a nonzero multiplicative linear functional on \( X \subset C(\Omega, X) \). The last two statements concerning \( \phi \) can be proved as follows: If \( \phi \) were identically zero on \( eC(\Omega) \) or \( X \), then \( \phi(xf) = \phi(ef) \cdot \phi(x) = 0 \) for all \( f \in C(\Omega) \) and all \( x \in X \). Since linear combinations of functions \( xf \) with \( f \in C(\Omega) \), \( x \in X \) are dense in \( C(\Omega, X) \) by Lemma 1, this would mean \( \phi \) is identically zero in \( C(\Omega, X) \) contrary to assumption.

It is easy to see that \( eC(\Omega) \) is isometrically isomorphic with \( C(\Omega) \). Hence, as is well known, there exists an \( a \in \Omega \) such that \( \phi(ef) = f(a) \) with \( f \in C(\Omega) \). Also there exists an \( M \in \mathfrak{M}(X) \) such that \( \phi(x) = \phi_M(x) \) for all \( x \in X \subset C(\Omega, X) \).

Suppose, now, that \( f \) is an arbitrary function in \( C(\Omega, X) \). By Lemma 1, there exists a sequence \( f_n \in C(\Omega, X) \) such that \( f_n \rightarrow f \) in \( C(\Omega, X) \)-norm
and furthermore $\phi(f) = \phi_M f(a)$. Since $\phi$ is continuous we have $\phi f = \phi_M f(a)$.

Suppose that $X$ lacks an $e$. Then we imbed $X$, isometrically and isomorphically, in a Banach algebra $X'$ with unit $e$ in such a way that the maximal ideals of $X'$ are the regular maximal ideals of $X$ and $X$ itself. This gives rise to an additional homomorphism of $X'$ onto the complex numbers, namely $\phi_X$, where $\phi_X(x) = 0$ if $x \in X$ and $\phi_X(\lambda x) = \lambda$ for all complex numbers $\lambda$. The space $\mathfrak{M}(X')$ is the one-point-compactification of $\mathfrak{M}(X)$ by $\phi_X$. By what we have already proved, the nonzero multiplicative functionals in $C(\Omega, X')$ are of the form $\phi_M f(a)$ and the additional functionals $\phi_X f(a)$. These latter functionals are all identically zero on $C(\Omega, X)$ so that the most general nonzero multiplicative functionals on $C(\Omega, X)$ are of the form $\phi_M f(a)$ with $M \in \mathfrak{M}(X)$, $a \in \Omega$. q.e.d.

**Corollary.** The only regular maximal ideals in $C(\Omega, X)$ are of the form \{ $f \in C(\Omega, X) \mid f(a) \in M$ \} with $a \in \Omega$, $M \in \mathfrak{M}(X)$.

**Theorem.** $\mathfrak{M}(C(\Omega, X))$ is homeomorphic with $\Omega \times \mathfrak{M}(X)$.

**Proof.** By the corollary, above, there is a 1-1 correspondence between the points of $\mathfrak{M}(C(\Omega, X))$ and those of $\Omega \times \mathfrak{M}(X)$. Now, the topology in $\mathfrak{M}(C(\Omega, X))$ is that induced by the family $\mathfrak{g}$ which consists of the functions $g_\ell(f) \in C(\Omega, X)$ defined on $\Omega \times \mathfrak{M}(X)$ by $g_\ell(a, M) = \phi_M f(a)$. We must show that this topology is identical with the product topology. For this purpose we use the following result (see [1, p. 12]):

If $\mathfrak{F}$ is a family of continuous complex-valued functions vanishing at infinity on a locally compact space $\Pi$, separating the points of $\Pi$ and not all vanishing at any point of $\Pi$, then the topology induced on $\Pi$ by $\mathfrak{F}$ is identical with the given topology of $\Pi$.

We take $\Pi = \Omega \times \mathfrak{M}(X)$ and define $\mathfrak{F}$ as follows. For each positive integer $n$ and each choice of $f_1, \ldots, f_n \in C(\Omega), x_1, \ldots, x_n \in X$, there is a function $h$ defined on $\Omega \times \mathfrak{M}(X)$ by $h(a, M) = \sum_{\ell=1}^n \phi_M(x_\ell f_\ell(a))$. Let $\mathfrak{F}$ be the family of all functions $h$ so defined. $\Pi$ is locally compact since $\Omega$ is compact and $\mathfrak{M}(X)$ is locally compact. Further, every function in $\mathfrak{F}$ is continuous in $(a, M)$ over $\Omega \times \mathfrak{M}(X)$ since this is true for the functions $\phi_M f(a)$ with $f \in C(\Omega, X)$ (see [2, Lemma 2.2]).

Not all functions in $\mathfrak{F}$ vanish at any point $(a_0, M_0) \in \Omega \times \mathfrak{M}(X)$. For, let $x \in X$ be such that $x \in M_0$ and pick $f \in C(\Omega)$ such that $f(a_0) \neq 0$. Then $\phi_M f(a_0) \neq 0$.

The functions in $\mathfrak{F}$ separate points in $\Omega \times \mathfrak{M}(X)$. Suppose $(a, M) \neq (b, N)$ with $a \neq b$. If $M = N$, take $x \in M$ and $f \in C(\Omega)$ such that
f(a) \neq f(b). Then \phi_M(xf(a)) \neq \phi_N(xf(b)). If M \neq N, find an x \in M, x \in N and f \in C(\Omega) such that f(a) = 0 and f(b) \neq 0. Then \phi_M(xf(a)) \neq \phi_N(xf(b)). If a = b so that M \neq N we may find x \in M, x \in N. Choose f \in C(\Omega) such that f(a) = f(b) \neq 0; then \phi_M(xf(a)) \neq \phi_N(xf(b)).

Finally we show that functions in \mathcal{F} vanish at \infty in \Omega \times \mathfrak{M}(X). Suppose \epsilon > 0 is given. If \sum_{i=1}^{n} \phi_M(x_i f_i(a)) \in \mathcal{F}, then \left| \sum_{i=1}^{n} f_i(a) \phi_M(x_i) \right| \leq \epsilon if (a, M) \in \Omega \times (U_{i=1} \mathcal{C}_i) where \left| \phi_M(x_i) \right| < \delta if M \in \mathcal{C}_i, \mathcal{C}_i being compact sets in \mathfrak{M}(X)(i = 1, 2, \cdots, n), and where \delta < \epsilon/nK with K = \sup_{i=1}^{n} \sup_{a \in \Omega} |f_i(a)|. The sets \mathcal{C}_i exist because each \phi_M(x_i) vanishes at infinity in \mathfrak{M}(X). Further, \Omega \times (U_{i=1} \mathcal{C}_i) is compact in \Omega \times \mathfrak{M}(X) so that each function in \mathcal{F} vanishes at \infty in \Omega \times \mathfrak{M}(X).

Using the result quoted in the beginning of the proof we have shown that the product topology of \Omega \times \mathfrak{M}(X) is identical with that induced by our family \mathcal{F}. Since \mathcal{F} is smaller than the family \mathcal{G} we see that the topology induced by \mathcal{G} on \Omega \times \mathfrak{M}(X) is stronger or equal to the topology induced by \mathcal{F}. Since the topology of \mathfrak{M}(C(\Omega, X)) is precisely that induced on \Omega \times \mathfrak{M}(X) by the family \mathcal{G}, the proof will be completed by showing that the \mathcal{F}- and \mathcal{G}-topologies on \Omega \times \mathfrak{M}(X) are identical. The family \mathcal{F} is contained in and is dense in \mathcal{G} in the uniform norm for continuous functions. For, suppose \epsilon \in C(\Omega, X). Then, by Lemma 1, we can find \sum_{i=1}^{n} x_i f_i, f_i \in C(\Omega), x_i \in X such that \left| f - \sum_{i=1}^{n} x_i f_i \right|_{C(a, X)} \leq \epsilon. This means \sup \left| \phi_M(a) - \sum_{i=1}^{n} \phi_M(x_i f_i(a)) \right| \leq \epsilon, where the sup is taken over all (a, M) \in \Omega \times \mathfrak{M}(X). Hence \mathcal{F} is dense in \mathcal{G} in the uniform norm and it follows that the \mathcal{F}- and \mathcal{G}-topologies are identical. q.e.d.

3. Concluding remarks. The essential ideas in this paper are direct outgrowths of the author's Yale doctoral thesis Group algebras of vector-valued functions. Just as in the thesis we can show, using the characterization of \mathfrak{M}(C(\Omega, X)) as \Omega \times \mathfrak{M}(X), that C(\Omega, X) is regular if and only if X is regular, and that every proper closed ideal in C(\Omega, X) is contained in a regular maximal ideal if X satisfies certain additional conditions. Further, results on kernels and hulls could be obtained in C(\Omega, X). All these can be proved by minor and obvious changes in the proofs of the thesis.

References

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