We define a class of commutative regular Banach algebras which contain simple examples of such algebras which are not self-adjoint. Conditions are given for the validity of the general Wiener theorem, and the denseness of certain translates. We use the Banach algebra terminology and results as presented in [1].

1. The Banach algebra $A$. Let $L^2(S)$ be the set of all complex-valued measurable functions, on a measure space $S$, whose magnitudes are square summable. We consider $L^2(S)$ as a Hilbert space, and assume the existence of a countable complete orthonormal set $\{\phi_k\}$ in $L^2(S)$, which has the additional property that $\phi_k \in L^\infty(S) \cap L^1(S)$ for all $k$. For the given $\{\phi_k\}$ let $\{v_k\}$ be any sequence of nonzero complex numbers satisfying the condition

$$\sum ||\phi_k||_{\infty}||\phi_k||_1 |v_k| \leq 1,$$

where $||\phi_k||_{\infty}$, $||\phi_k||_1$ denote the norms of $\phi_k$ in $L^\infty(S)$ and $L^1(S)$ respectively. If $f, g \in L^1(S)$ we define the product $f \ast g$ by

$$f \ast g = \sum (f, \phi_k)(g, \phi_k)v_k\phi_k.$$

Here $(f, \phi_k) = \int_S f\phi_k^*$, and in general $(f, \alpha) = \int_S f\alpha^*$ in case $f \in L^1(S)$, $\alpha \in L^\infty(S)$, or in case $f, \alpha \in L^2(S)$. By (1), the series converges in $L^1(S)$, and indeed

$$||f \ast g||_1 \leq ||f||_1||g||_1 \quad (f, g \in L^1(S)).$$

We have the following result:

$L^1(S)$ with $\ast$ as multiplication is a commutative Banach algebra $A$ which is regular.

**Proof.** From (2) it follows that $A$ is closed under multiplication and satisfies the norm requirement for the product. The definition of the product clearly shows that the commutative and associative laws for multiplication are valid. Thus $A$ is a commutative Banach algebra.

It remains to prove that $A$ is regular, and for this we determine the regular maximal ideal space $\mathfrak{M}$ for $A$. The points $M$ of $\mathfrak{M}$ are in a one-to-one correspondence with the nonzero continuous homomorphisms of $A$ onto the complex numbers. Let $l_M$ be such a homo...
morphism corresponding to \( M \in \mathfrak{M} \). Then \( l_M(\phi_k) \) cannot be zero for all \( k \), for if the contrary is true, we have for any \( f, g \in L^1(S) \),

\[
l_M(f)l_M(g) = l_M(f \ast g) = \sum (f, \phi_k)(g, \phi_k)\nu_kl_M(\phi_k) = 0.
\]

Since \( l_M \) can not be identically zero, this gives a contradiction. Let \( k \) be such that \( l_M(\phi_k) \neq 0 \). Then \( l_M(f)l_M(\phi_k) = l_M(f \ast \phi_k) = (f, \phi_k)\nu_kl_M(\phi_k) \), or \( l_M(f) = (f, \nu_k\phi_k) \). There is exactly one \( k \) such that \( l_M(\phi_k) \neq 0 \), for \( l_M(\phi_j) = (\phi_j, \nu_k\phi_k) = 0 \), if \( j \neq k \). Thus corresponding to \( M \in \mathfrak{M} \) there is a unique integer \( k \) such that \( l_M(f) = (f, \nu_k\phi_k) \). Conversely every \( \nu_k\phi_k \) generates in this way a continuous homomorphism onto the complex numbers, and the \( f \in A \) satisfying \( (f, \nu_k\phi_k) = 0 \) form a regular maximal ideal \( M \). Therefore \( \mathfrak{M} \) can be identified with the integers \( M \leftrightarrow k \).

For any \( f \in A \) let \( \hat{f} \) be the complex-valued function defined on \( \mathfrak{M} \) by \( \hat{f}(M) = \hat{f}(k) = (f, \nu_k\phi_k) \). Then \( A \) is said to be regular if, given any closed set \( C \) in \( \mathfrak{M} \) and point \( M_0 \) not in \( C \), there exists an \( f \in A \) such that \( \hat{f}(M) = 0 \) on \( C \), \( \hat{f}(M_0) \neq 0 \). For our \( A \) let \( C \) be any set in \( \mathfrak{M} \) and \( M_0 \) a point not in \( C \) determined by the integer \( k \). Then \( \phi_k \in A \) has the property that \( \hat{\phi}_k(j) = \delta_{jk}\nu_j \), thus showing that \( A \) is regular.

The algebra \( A \) is semi-simple (the map \( f \rightarrow \hat{f} \) is one-to-one) if and only if \( (f, \phi_k) = 0 \) for all \( k \) implies \( f = 0 \) almost everywhere. Suppose \( A \) is semi-simple. If \( f \in A \), and \( \hat{f} \) vanishes outside a compact set, then \( (f, \phi_k) = 0 \) for \( |k| > N \), for some positive integer \( N \). Thus

\[
f = \sum_{|k| \leq N} (f, \phi_k)\phi_k,
\]

since \( (f - \sum_{|k| \leq N} (f, \phi_k)\phi_k, \phi_j) = 0 \) for all \( j \). It follows that the set of elements \( f \in A \), such that \( \hat{f} \) vanishes outside a compact set, is dense in \( A \) if and only if the sequence \( \{ \phi_k \} \) is dense in \( A \). The general Wiener Tauberian theorem (in the form of the Corollary, p. 85, [1]) then takes the following form.

Suppose the sequence \( \{ \phi_k \} \), which defines the algebra \( A \), satisfies the two conditions:

(a) \( (f, \phi_k) = 0 \), for \( f \in L^1(S) \) and all \( k \), implies \( f = 0 \) almost everywhere,

(b) \( \{ \phi_k \} \) is dense in \( L^1(S) \).

Then every proper closed ideal is included in a regular maximal ideal.

2. Some examples. A commutative Banach algebra \( A \) is said to be self-adjoint if for every \( f \in A \) there exists a \( g \in A \) such that \( \hat{g} = f \). We give an example of a choice of \( \{ \phi_k \} \) and \( \{ \nu_k \} \) which lead to an algebra which is regular, but not self-adjoint. Let \( S \) be the real interval \( -\pi \leq x \leq \pi \), and \( \phi_k = (2\pi)^{-1/2}e^{-ikx} \), \( k = 0, \pm 1, \pm 2, \ldots \). Clearly \( \phi_k \in L^\infty(S) \cap L^1(S) \cap L^2(S) \). The condition for self-adjointness is that for each \( f \in L^1(S) \) there exists a \( g \in L^1(S) \) such that \( \hat{g}(k) = (g, \nu_k\phi_k) \).
\[(g, \phi_k) = e^{2i\theta_k}(f, \phi_k)^c. \]

Let \(f^c(x) = [f(-x)]^c\). Clearly \(f^c \in L^1(S)\) if and only if \(f \in L^1(S)\), and \((f, \phi_k)^c = (f^c, \phi_k)\). Thus \(L^1(S)\) is self-adjoint if and only if, given any \(f^c \in L^1(S)\), there exists a \(g \in L^1(S)\) such that

\[(g, \phi_k) = e^{2i\theta_k}(f^c, \phi_k) \quad (k = 0, \pm 1, \cdots).\]

Now the function \(f^c\) defined by

\[f^c(x) = \sum_{k=1}^{\infty} k^{-1/4}e^{ikx}\]

is in \(L^1(-\pi, \pi)\), but a sequence of \(\pm\) signs exist for which

\[\sum_{k=1}^{\infty} \pm k^{-1/4}e^{ikx}\]

is not the Fourier series of any function in \(L^1(-\pi, \pi)\) (see [2, p. 212]). If we choose the \(\theta_k\) so that \(e^{2i\theta_k}\) gives such a sequence of signs, we see that \(A\) will not be self-adjoint. Note that this \(A\) is semi-simple and satisfies the conditions (a) and (b) for the general Wiener theorem.

Further examples are obtained by letting the \(\phi_k\) be the orthonormal eigenfunctions of a self-adjoint ordinary differential operator on a finite closed interval \(S: a \leq x \leq b\). The \(\phi_k\) are continuous, and it is known that both (a) and (b) are valid.

Certain singular self-adjoint differential operators have a pure point spectrum. For example, the problem \(-u'' + q(x)u = \lambda u\), on \(S: -\infty < x < \infty\), is self-adjoint and has a pure point spectrum if \(q\) is real-valued, continuous, and \(q(x) \to \infty\) as \(x \to \pm \infty\). It is known that in this case the orthonormal eigenfunctions \(\phi_k\) are in \(L^\infty(S) \cap L^1(S) \cap L^2(S)\). For the particular case where \(q(x) = x^2\) the \(\phi_k\) are the Hermite functions, and the conditions (a) and (b) are valid.

3. The denseness of certain translates. For each \(y \in S\) let \(T_y\) be defined by

\[T_yf(x) = \sum (f, \phi_k)^c \nu^c_k[\phi_k(y)]^c \phi_k.\]

From (1) it follows that \(\|T_yf\|_1 \leq \|f\|_1\) for every \(f \in L^1(S)\), and an equivalent definition of \(f \star g\) is

\[f \star g(x) = \int g(y)dy.\]

We call \(T_yf\) the translate of \(f\) by \(y\). Let \(\mathfrak{T}_f\), for a given \(f \in L^1(S)\), be the set of all finite linear combinations, with complex coefficients, of
translates of $f$. It does not appear obvious that the closure of $\mathcal{I}_f$ is an ideal in $A$, in case $A$ satisfies (a) and (b). Nevertheless it is easy to show directly the following result:

Suppose $\{\phi_k\}$ is dense in $A$, and $f \in A$ is such that $(f, \phi_k) \neq 0$ for any $k$. Then $\mathcal{I}_f$ is dense in $A$.

Proof. By the Hahn-Banach theorem $\mathcal{I}_f$ is dense in $A$ if and only if the only bounded linear functional $l$ which satisfies $l(g) = 0$ for all $g \in \mathcal{I}_f$ is the identically zero one. Suppose $l(g) = 0$ for all $g \in \mathcal{I}_f$. In particular $l(T_{\phi}) = 0$ for all $y \in S$. Thus, using the definition (3),

$$\sum (f, \phi_k) v_k [\phi_k(y)] l(\phi_k) = 0 \quad (y \in S),$$

where the series converges in $L^\infty(S)$ by (1). Since $\phi_j \in L^1(S)$ we have $0 = (\phi_j, 0) = (f, \phi_j) v_j l(\phi_j)$, and, since $(f, \phi_j) v_j \neq 0$ for any $j$, this implies $l(\phi_j) = 0$ for all $j$. This in turn implies that $l$ is the identically zero linear functional, since the $\phi_j$ are dense in $A$.

References


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