ISOMORPHISM OF AW*-ALGEBRAS

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We shall prove in this note

THEOREM. *Let M, N be two AW^*-algebras having no component of type I_n $(n=1, 2, \cdots)$. Let ρ be an algebraic isomorphism of the unitary group M_u of M onto the unitary group N_u of N. Then ρ induces an orthoisomorphism θ between the projection lattices M_P and N_P of respectively M and N.*

H. A. Dye, in his paper *On the geometry of projections in certain operator algebras*,[1] has shown that the structure of the projection lattice of an AW*-algebra determines the structure of the algebra. He then goes on to show that, in the case of a factor not of type I_{2n}, the structure of projection lattice is determined by the unitary structure. The crucial step in his proof lies in the study of the family of unitary operators $\rho(\lambda e+1-e)$ where e is a fixed projection and λ takes on all complex numbers of absolute value 1. Following his method by examining the operators $\rho(\lambda e+1-e)$ in greater detail we are able to extend his result. However, since we rely heavily on the existence of four orthogonal equivalent projections, we are not able to get any information about algebras of type I_{2n+1}.

We first collect a few results about AW*-algebras which are pertinent to our discussion.

(a) If e, f are orthogonal equivalent projections then there exists a unitary operator u such that $f=ueu^{-1}$ and $u(1-e-f)=(1-e-f)u=1-e-f$. In fact, if v is a partial isometry with $vv^*=e$, $v^*v=f$, then $u=v+v^*+(1-e-f)$ will do.

(b) A projection e is called properly infinite if, for every central projection g, ge is either infinite or zero. Given any projection e there is a central projection g such that ge is properly infinite (or zero) and $(1-g)e$ is finite.

(c) By the central cover of a projection e we mean the least central projection containing e. Let e be properly infinite and f finite. Then, if e, f have the same central cover, e contains infinitely many orthogonal projections equivalent to f.

Let M, N be AW*-algebras having no component of type I_n, ρ an algebraic isomorphism between the unitary groups M_u and N_u. De-

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[1] Ann. of Math. vol. 61 (1955) pp. 73–89. We assume that the reader is familiar with this paper.

fine θ by the equation $\rho(1-2e)=1-2\theta(e)$, where e is a projection. Then θ is a one-to-one map of M_P onto N_P. θ has the following properties:

(d) $\theta(ueu^{-1})=\rho(u)\theta(e)\rho(u)^{-1}$ for any $e\in M_P$, $u\in M_u$.

(e) θ preserves commutativity.

(f) θ preserves the symmetric difference of commuting projections: $\theta(e\Delta f)=\theta(e)\Delta\theta(f)$, where $e\Delta f=e+f-2ef$.

LEMMA 1. *For a fixed projection e there exist two homomorphic maps u, v of the multiplicative group of complex numbers of absolute value* 1 *into the center Z_N of N such that*

$$\rho(\lambda e+1-e)=u(\lambda)\theta(e)+v(\lambda)(1-\theta(e)) \tag{1}$$

and that $u(\lambda)u(\lambda)^*=$central cover of $\theta(e)$, $v(\lambda)v(\lambda)^*=$central cover of $1-\theta(e)$.

PROOF. Let $N(e)$ denote the AW*-subalgebra generated by $\rho(\lambda e+1-e)$. $N(e)$ is contained in the intersection of all commutative AW*-subalgebras which contain $\rho(\lambda e+1-e)$. Thus $\rho(u)$ $(u\in M_u)$ commutes with elements of $N(e)$ if it commutes with $\rho(\lambda e+1-e)$. $\rho(u)$ commutes with $\rho(\lambda e+1-e)$ if and only if u commutes with e. Hence $\rho(u)$ commutes with $\rho(\lambda e+1-e)$ if and only if $\rho(u)$ commutes with $\theta(e)$. Therefore $\theta(e)N(e)$ is contained in the center $\theta(e)Z_N$ of $\theta(e)N\theta(e)$. Similarly $(1-\theta(e))N(e)$ is contained in the center $(1-\theta(e))Z_N$ of $(1-\theta(e))N(1-\theta(e))$. Thus we have

$$N(e)=\theta(e)N(e)+(1-\theta(e))N(e)\subseteq\theta(e)Z_N+(1-\theta(e))Z_N.$$

Consequently

$$\rho(\lambda e+1-e)=u(\lambda)\theta(e)+v(\lambda)(1-\theta(e)), \tag{1}$$

where $u(\lambda)$, $v(\lambda)$ are in Z_N. The properties of u, v can be checked easily. This completes the proof.

We shall also use $u(\lambda, e)$, $v(\lambda, e)$ for $u(\lambda)$, $v(\lambda)$ to indicate their dependence on e.

LEMMA 2. *Let e, f be orthogonal equivalent projections. Then $\theta(e)\theta(f)=g(1-\theta(e+f))=g((1-\theta(e))\Delta\theta(f))$, where g is the central cover of $\theta(e)\theta(f)$.*

PROOF. Since e, f are unitarily equivalent $u(\lambda, e)=u(\lambda, f)=u(\lambda)$ and $v(\lambda, e)=v(\lambda, f)=v(\lambda)$. Apply (1) to e, f and $h=e+f$ we get

$$\rho(\lambda e+1-e)=u(\lambda)\theta(e)+v(\lambda)(1-\theta(e)),$$

$$\rho(\lambda f+1-f)=u(\lambda)\theta(f)+v(\lambda)(1-\theta(f)),$$

$$\rho(\lambda h+1-h)=u(\lambda, h)\theta(h)+v(\lambda, h)(1-\theta(h)).$$

Since

$$\lambda h + 1 - h = (\lambda e + 1 - e)(\lambda f + 1 - f),$$

$$u(\lambda, h)\theta(h) + v(\lambda, h)(1 - \theta(h))$$
$$= u(\lambda)v(\lambda)\theta(h) + u(\lambda)^2\theta(e)\theta(f) + v(\lambda)^2(1 - \theta(e))(1 - \theta(f)).$$

It follows that

$$v(\lambda, h) = u(\lambda^2) \text{ on the central cover } g \text{ of } \theta(e)\theta(f),$$
$$v(\lambda, h) = v(\lambda)^2 \text{ on the central cover } g_1 \text{ of } (1 - \theta(e))(1 - \theta(f)).$$

Therefore

$$u(\lambda^2) = v(\lambda)^2 \text{ on } gg_1.$$

Since $1-2\theta(e)=\rho(-e+1-e)$, $u(-1)=-k$, $v(-1)=k_1$, where k, k_1 are respectively the central cover of $\theta(e)$ and $1-\theta(e)$. $gg_1<k, k_1$. Hence

$$-gg_1 = u(-1)gg_1 = u(i)^2gg_1 = v(i)^2gg_1 = gg_1.$$

Consequently $gg_1=0$ and $\theta(e)\theta(f)=g(1-\theta(e+f))$.

LEMMA 3. *Let e_1, e_2, f_1, f_2 be orthogonal equivalent projections, $e=e_1+e_2$ $f=f_1+f_2$. Then $\theta(e)\theta(f)=0$.*

PROOF. By (a) $\theta(e_1)\theta(e_2)$, $\theta(e_1)\theta(f_1)$, $\cdots$, $\theta(f_1)\theta(f_2)$ are equivalent projections; let g be their common central cover.

$$\begin{aligned} g\theta(f_1)\theta(e) &= g\theta(f_1)(\theta(e_1)\Delta\theta(e_2)) \\ &= g(\theta(f_1)\theta(e_1)\Delta\theta(f_1)\theta(e_2)) \\ &= g[(1 - \theta(f_1))\Delta\theta(e_1)\Delta(1 - \theta(f_1))\Delta\theta(e_2)] \\ &= g(\theta(e_1)\Delta\theta(e_2)) = g\theta(e), \end{aligned}$$

since $1-\theta(f_1)$ appears twice. Therefore

$$\begin{aligned} g\theta(e) &= g\theta(f_1)\theta(f_2)\theta(e) = g(1 - \theta(f))\theta(e) \\ &= g\theta(e) - g\theta(e)\theta(f), \end{aligned}$$

and $g\theta(e)\theta(f)=0$. Now the central cover of $\theta(e)\theta(f)$ is contained in g for $\theta(e)\theta(f)=[\theta(e_1)+\theta(e_2)-2\theta(e_1)\theta(e_2)][\theta(f_1)+\theta(f_2)-2\theta(f_1)\theta(f_2)]$. Consequently $\theta(e)\theta(f)=0$.

LEMMA 4. *Let e, f be orthogonal projections such that each is a sum of four orthogonal equivalent projections. Then $\theta(e)\theta(f)=0$.*

PROOF. Let $e=e_1+\cdots+e_4$, $f=f_1+\cdots+f_4$, where $e_1, \cdots, e_4$ and $f_1, \cdots, f_4$ are two families of orthogonal equivalent projections. Let $h_i=e_i+f_i$, $i=1, \cdots, 4$. By Lemma 3

$$\theta(e_1 + e_2)\theta(e_3 + e_4) = \theta(f_1 + f_2)\theta(f_3 + f_4)$$
$$= \theta(h_1 + h_2)\theta(h_3 + h_4) = 0.$$

But

$$\theta(h_1 + h_2)\theta(h_3 + h_4) = (\theta(e_1 + e_2) + \theta(f_1 + f_2) - 2\theta(e_1 + e_2)\theta(f_1 + f_2))$$
$$\cdot(\theta(e_3 + e_4) + \theta(f_3 + f_4) - 2\theta(e_3 + e_4)\theta(f_3 + f_4))$$
$$= \theta(e_1 + e_2)\theta(f_3 + f_4) + \theta(e_3 + e_4)\theta(f_1 + f_2).$$

Therefore

$$\theta(e_1 + e_2)\theta(f_1 + f_2) = \cdots$$
$$= \theta(e_3 + e_4)\theta(f_1 + f_2) = \theta(e_3 + e_4)\theta(f_3 + f_4) = 0.$$

It follows that $\theta(e)\theta(f) = 0$.

Proof of the theorem:

(i) Since M has no component of type I_n, each central projection is a sum of four orthogonal equivalent projections. By Lemma 4, θ sends orthogonal central projections into orthogonal central projections. Consequently, if $\theta(h) = 1$ then $h = 1$, i.e., $\theta(1) = 1$.

(ii) Hereafter in the proof h denotes the type I component of the identity of M and $g = 1 - h$. Then every projection in g is a sum of four orthogonal equivalent projections. Therefore θ sends orthogonal projections in g into orthogonal projections. Moreover, $\theta(e) < \theta(g)$ if $e < g$.

(iii) Let e be a properly infinite projection in h. Split e into a sum of two orthogonal equivalent projections e_1, e_2. Then e_1, e_2 are properly infinite and, therefore, each is a sum of four orthogonal equivalent projections. Then $\theta(e_1)\theta(e_2) = 0$ by Lemma 4 and $\theta(e) = \theta(e_1) + \theta(e_2)$. Now $h - e_1$, containing e_2, is properly infinite. Therefore $\theta(h - e_1)\theta(e_1) = 0$. It follows that $\theta(e_1) < \theta(h)$ and $\theta(e) < \theta(h)$.

(iv) Let e be a finite projection in h. By (c) $h - e$ contains a sequence of orthogonal projections $e_1, e_2, \cdots$, each equivalent to e. Write $f_1 = \sum_{i=1}^{\infty} e_{2i-1}$, $f_2 = \sum_{i=1}^{\infty} e_{2i}$. $\theta(f_2)$ is orthogonal to $\theta(f_1)$ and $\theta(f_1 + e)$. Therefore $\theta(f_2)$ is orthogonal to $\theta(e)$ since $\theta(e) = \theta(f_1 + e) - \theta(f_1) + 2\theta(f_1)\theta(e)$. By (iii) $\theta(f_2)$ and $\theta(f_2 + e)$ are contained in $\theta(h)$, therefore $\theta(e)$ is contained in $\theta(h)$.

(v) Let e be a projection, k its central cover. It follows from (ii)–(iv) that $\theta(e) < \theta(k)$. Hence to show that θ sends orthogonal projections into orthogonal projections, it remains to examine the behavior of θ on projections contained in h. If two projections are separated by orthogonal central projections so are their image under θ. We may, therefore, restrict our consideration to projections which are

properly infinite or finite and which have the same central cover. The case of two properly infinite projections is treated in Lemma 4.

(vi) Let e, f be finite orthogonal projections having the same central cover $k<h$. $k-(e+f)$ contains infinitely many orthogonal projections, each equivalent to $e+f$. Consequently, there exist orthogonal projections $e_1, f_1, e_2, f_2, \cdots$ in $k-(e+f)$ such that $e_i \sim e$, $f_i \sim f$. We have shown in (iv) that $\theta(e)$ is orthogonal to $\theta(\sum_{i=1}^{\infty} e_i)$ and $\theta(\sum_{i=2}^{\infty} e_i)$. Therefore $\theta(e)$ is orthogonal to $\theta(e_1)$. Similarly $\theta(f)$ is orthogonal to $\theta(f_1)$. Then, by Lemma 4, $\theta(e)+\theta(e_1)+\theta(e_2)+\theta(e_3)=\theta(e+\cdots+e_3)$ is orthogonal to $\theta(f)+\theta(f_1)+\theta(f_2)+\theta(f_3)=\theta(f+\cdots+f_3)$. It follows that $\theta(e)$ is orthogonal to $\theta(f)$.

(vii) Let e, f be orthogonal projections having the same central cover $k<h$. Suppose that e is finite, f properly infinite. Split f into a sum of two equivalent orthogonal projections: $f=f_1+f_2$. $\theta(f_1)$ is orthogonal to $\theta(f_2)$ and $\theta(f_2+e)$ by Lemma 4. Hence $\theta(f_1)$ is orthogonal to $\theta(e)$. Likewise $\theta(f_2)$ is orthogonal to $\theta(e)$, whence $\theta(f)=\theta(f_1)+\theta(f_2)$ is orthogonal to $\theta(e)$.

(viii) Every central projection in N is of the form $\theta(h)$ where h is a central projection in M. Therefore N cannot have a component of type I_n if M does not. Hence θ^{-1} also preserves orthogonality and θ is an orthoisomorphism of the projection lattices M_P and N_P.

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