COUNTEREXAMPLES TO THE POINCARÉ INEQUALITY

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Given a bounded plane domain D, the Poincaré inequality can be stated as follows (see for example [1, Chapter VII]): *there exists a constant K such that for any function ϕ having a finite Dirichlet integral over D and belonging to a class $\mathfrak{F}$*

$$\iint_D \phi^2 dxdy \leqq K \iint_D (\phi_x^2 + \phi_y^2) dxdy. \tag{1}$$

The inequality (1) holds true when $\mathfrak{F}$ is the class of all continuous real functions which are zero on the boundary of D. It also holds when $\mathfrak{F}$ is the class of all continuous real functions such that

$$\iint_D \phi dxdy = 0, \tag{2}$$

provided that the domain D is sufficiently regular. It suffices in this case that D be decomposible into a finite number of convex regions.

In [1, p. 521], Courant and Hilbert give an example which shows that (1) does not hold under the normalization (2) unless some such restriction is placed on the domain D. In the example which they consider, the domain D has a Jordan curve as its boundary, but the function ϕ is quite irregular.

In this note, an example is given (Theorem 1) which shows that (1) does not hold under the normalization (2) even if ϕ is restricted to be harmonic. The domain D considered is simply connected and bounded, but possesses a nontrivial prime end as part of its boundary.

When treating analytic functions, it is usually more convenient to fix a point z_0 in D, and normalize the functions by demanding $f(z_0) = 0$. The Poincaré inequality then becomes

$$\iint_D |f(z)|^2 dxdy \leqq K \iint_D |f'(z)|^2 dxdy.$$

It appeared at first thought that the Poincaré inequality must hold for any bounded domain in this case, but the same method also shows that this is not true (Theorem 2).

The examples used seem to be notable for their simplicity. In the

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w-plane, with $w=u+iv$, define a domain Δ by

$$\Delta = \{ w: u_1 < u < \infty, f_1(u) < v < f_2(u) \} \tag{3}$$

where f_1 and f_2 are continuous, bounded, and decreasing in $u_1 \leqq u < \infty$. We can assume that f_1 and f_2 are positive and tend to zero as $u \to \infty$. Furthermore we demand that for any u, $u_1 \leqq u < \infty$

$$0 < f_2(u + 2\pi) \leqq f_1(u) < f_2(u).$$

Under the mapping

$$z = e^{iw} \tag{4}$$

Δ is then transformed one-to-one onto a simply connected plane domain D, interior to the unit circle. In particular, if $f_1(u) = f_2(u+2\pi)$ for all $u \geqq u_1$, then the domain D has as its boundary a spiral of infinite length inside the unit circle, a short radial segment joining two turns of this spiral, and the entire unit circle (as a prime end).

Finally we note that if $f_2(u_1) = M$, i.e., $f_2(u) < M$ for all $u > u_1$, then since $|dz/dw| = e^{-v}$, for any w in Δ

$$e^{-2M} < \left| \frac{dz}{dw} \right|^2 < 1. \tag{5}$$

THEOREM 1. *There exists a bounded, simply connected domain D in the z-plane and a real function ϕ, harmonic in D such that*

$$\iint_D \phi dx dy = 0, \qquad \iint_D (\phi_x^2 + \phi_y^2) dx dy < \infty,$$

and

$$\iint_D \phi^2 dx dy = \infty.$$

PROOF. Define the domain Δ by (3), taking $f_2(u) = 1/u^2$, $f_1(u) = f_2(u+2\pi)$, and $u_1 = 1$. Then D is a bounded, simply connected domain. Define

$$\phi(x, y) = u - c$$

where $w = u + iv \leftrightarrow z = x + iy$ under the mapping (4). This function is then harmonic in D. The constant c is to be so chosen that

$$\iint_D \phi dx dy = 0.$$

In view of (5), this may be done and the theorem proved if we can show

$$\iint_\Delta dudv < \infty, \qquad \iint_\Delta ududv < \infty, \qquad \iint_\Delta u^2dudv = \infty.$$

But by simple computation

$$\iint_\Delta dudv = \int_1^\infty du \int_{1/(u+2\pi)^2}^{1/u^2} dv = 4\pi \int_1^\infty \frac{(u+\pi)}{u^2(u+2\pi)^2}\, du < \infty,$$

$$\iint_\Delta ududv = 4\pi \int_1^\infty \frac{(u+\pi)}{u(u+2\pi)^2}\, du < \infty,$$

while

$$\iint_\Delta u^2dudv = 4\pi \int_1^\infty \frac{(u+\pi)}{(u+2\pi)^2}\, du = \infty.$$

The theorem then follows immediately from these relations.

THEOREM 2. *There exists a bounded, simply connected domain D in the z-plane, containing a point z_0, and an analytic function $f(z)$ in D such that*

$$f(z_0) = 0, \qquad \iint_D |f'(z)|^2 dxdy < \infty,$$

while

$$\iint_D |f(z)|^2 dxdy = \infty.$$

PROOF. We may use the same domain D as defined in the proof of Theorem 1 (or we could set $f_2(u) = 1/u$ for this case). If $z_0 \leftrightarrow w_0 = u_0 + iv_0$, define $f(z) = w - w_0$. Now if $\operatorname{Re} w > 2u_0$, then $|w - w_0| > u/2$. Hence

$$\iint_D |f'(z)|^2 dxdy = \iint_\Delta dudv < \infty,$$

while

$$\iint_D |f(z)|^2 dxdy = \iint_\Delta |w - w_0|^2 \left|\frac{dz}{dw}\right|^2 dudv$$

$$\geqq \pi e^{-2} \int_{2u_0}^\infty \frac{(u+\pi)}{(u+2\pi)^2}\, du = \infty.$$

This proves the theorem.

The writer is indebted to Professor P. R. Garabedian for suggesting that a counterexample of this type must exist.

Reference

1. Courant and Hilbert, *Methoden der mathematischen Physik*, vol. 2, 1937.

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ON THE ARTIN-HASSE EXPONENTIAL SERIES

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1. Professor G. Whaples has kindly drawn my attention to the very similar properties enjoyed by the series which I called the Witt hyperexponential in a recent paper [2], and a series which he had previously defined, using the Artin-Hasse exponential series [5]; the main fact is that both series define a homomorphism of the Witt group W onto the multiplicative group W_1^*. In answer to his questions, I propose in this note to clear up completely that relationship, by determining *all* formal power series which define such homomorphisms, in other words, what one might call the formal *characters* of the group W; it turns out that the Artin-Hasse-Whaples series is the simplest member of that family, from which all others can be deduced by a simple transformation. I am indebted to Professor Whaples for several useful remarks and comments, as well as for pointing out a slight error in one of my original proofs.

2. Let $(a_0, a_1, \cdots, a_i, \cdots)$ be an infinite sequence of rational numbers, and let us consider the power series in one indeterminate x

$$(1) \qquad \exp(a_0x + a_1x^p + a_2x^{p^2} + \cdots + a_ix^{p^i} + \cdots) = \sum_{n=0}^{\infty} c_nx^n$$

where p is a prime number.

Proposition 1. *In order that in the series* (1) *all coefficients* c_n *be* p-*adic integers, a necessary and sufficient condition is that, for each* $i \geqq 0$, *one should have*

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