A GENERALIZED LAPLACE-STIELTJES TRANSFORMATION

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1. Introduction. Gustav Doetsch\(^1\) has recently introduced a generalized Laplace transform of order \(k, k \geq 0\). It is the object of this paper to introduce a generalized Laplace-Stieltjes transform of order \(k\).

We assume in what follows that \(k\) is positive and integral. We also assume that \(F\) denotes a complex-valued function of a real variable, which is defined, except perhaps on a set of isolated points, and is of bounded variation on every closed, finite interval \([0, T]\), \(T \geq 0\). By \(\{F\}\) we mean the class of all functions \(F\) defined above. All integrals in this paper are to be interpreted as Riemann or Riemann-Stieltjes integrals. The symbol (*) will denote a convolution.

2. Definition of the transform. Let \(s\) be complex. Let \(M_k(s, t)\) be the Cesàro mean of order \(k\) of

\[
\int_0^t e^{-su}dF(u),
\]

i.e.,

\[
M_k(s, t) = \frac{k}{t^k} \left\{ \left[ \int_0^t e^{-su}dF(u) \right] * t^{k-1} \right\}.
\]

If for some \(s\) and some \(k\)

\[
\lim_{t \to +\infty} M_k(s, t)
\]

exists, then this limit defines a value \(f_k(s)\) of a function \(f_k\):

\[
f_k(s) = \lim_{t \to +\infty} M_k(s, t).
\]

We call \(f_k\) the generalized Laplace-Stieltjes transform of order \(k\) of \(F\).

3. Existence and integral representations of the transform. We assume in what follows that \(s_0\) is a fixed complex number. From the definition of \(f_k(s_0)\) and the consistency of Cesàro summability we can assert the following:

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Theorem 1. If $f_k(s_0)$ exists and $N$ is a nonnegative integer, then, as $t \to +\infty$,
\[
\left[ \int_0^t e^{-su}dF(u) - f_k(s_0) \right] * t^{k+N-1} = o(t^{k+N}).
\]

We now define
\[
(1) \quad H_i(t) = k \left[ \int_0^t e^{-su}dF(u) - f_k(s_0) \right] * t^{k-2+i}, \quad i = 1, 2.
\]

Then, by Theorem 1, if $f_k(s_0)$ exists,
\[
(2) \quad H_i(t) = o(t^{k-1+i}), \quad i = 1, 2,
\]
as $t \to +\infty$. We also define
\[
(3) \quad G(t) = \int_0^t e^{-su}dF(u),
\]
so that we may now write (1) as
\[
(4) \quad H_i(t) = k \left[ G(t) - f_k(s_0) \right] * t^{k-2+i}, \quad i = 1, 2.
\]

In what follows we define an angular region $Q_w(s_0)$ to be the totality of all $s$ such that $|\arg (s-s_0)| \leq w$, where $w$ is a fixed number satisfying $0 \leq w < \pi/2$.

Using the facts that $|s-s_0| \leq R(s-s_0)/\cos w$ for every $s$ in $Q_w(s_0)$ \(-\{s_0\}\) and that for every $\epsilon_0 > 0$ there exists, by (2), a $T_i$ such that $|H_i(t)| < \epsilon_0 t^{k-1+i} (\cos^{k-1+i} w)/(k-1+i)!$ whenever $t > T_i$, we can establish the following theorem:

Theorem 2. Let $f_k(s_0)$ exist. Let $Q_w(s_0)$ be any fixed angular region. Then for any $\epsilon_0 > 0$ and nonnegative integer $h$ a $T_i$ exists such that uniformly in $Q_w(s_0)$
\[
|(s-s_0)^{k-1+i}e^{-(s-s_0)t}H_i(t)| < \epsilon_0 t^{k-h}
\]
whenever $t > T_i$, $i = 1, 2$.

Theorem 3. If $f_k(s_0)$ exists and $R(s) > R(s_0)$, then $f_k(s)$ exists. Furthermore, $M_k(s, t)$ converges uniformly to $f_k(s)$ in every region $Q_w(s_0)$. If $R(s) > R(s_0)$ or if $s = s_0$, the following representations of $f_k(s)$ are valid:
\[
(f_k(s) = \int_0^\infty \frac{(s-s_0)^{k+2}}{k!} e^{-(s-s_0)t} \left\{ \left[ \int_0^t e^{-su}dF(u) - f_k(s_0) \right] * t^k \right\} dt + f_k(s_0),
(5)
\]
The integrals in (5) and (6) are uniformly convergent in every region \( Q_w(s_0) \). The integral in (5) is absolutely convergent for \( R(s) > R(s_0) \).

Proof. Assertions of the theorem for the case \( s = s_0 \) can be easily verified. We therefore assume \( s \neq s_0 \). Reflection now reveals that it is immaterial whether we consider \( s \) to be such that \( R(s) > R(s_0) \) or whether we consider it to be such that it lies in some region \( Q_w(s_0) \setminus \{ s_0 \} \). We therefore further assume that \( s \) is in some region \( Q_w(s_0) \setminus \{ s_0 \} \).

By Theorem 6b of Widder\(^2\) and (3) we may write, since \( R(s) > R(s_0) \),

\[
\int_0^t e^{-su}dF(u) = \int_0^t e^{-s-u}dG(u).
\]

From this it follows that

\[
M_k(s, t) = \frac{k}{t^k} \int_0^t \left\{ \int_0^u e^{-s-u}dG(v) \right\} (t - u)^{k-1}du,
\]

or, since the Stieltjes integral with respect to a constant is 0,

\[
M_k(s, t) = \frac{k}{t^k} \int_0^t \left\{ \int_0^u e^{-s-u}d[G(v) - f_k(s_0)] \right\} (t - u)^{k-1}du.
\]

After integrating by parts, using a method of Doetsch's\(^3\) and referring to (4) we get

\[
M_k(s, t) = \frac{e^{-s-t}e^{-s-t}}{t^k} H_1(t) + \frac{(s - s_0)e^{-s-t}}{kt^k} H_2(t)
\]

\[+ \frac{1}{t^k} \int_0^t I_1(u, t)e^{-s-u}H_1(u)du \]

\[+ \frac{1}{kt^k} \int_0^t I_2(u, t)e^{-s-u}H_2(u)du \]

\[+ \frac{1}{k(k+1)} \int_0^t (s - s_0)^{k+1}e^{-s-u}H_2(u)du + f_k(s_0),\]

where


\(^{3}\) Doetsch, op. cit., pp. 316–317.
\[ I_i(u, t) = \sum_{h=1}^{k-1+i} \binom{k-1+i}{h} \frac{(s - s_0)^{k-1+i}}{(h-1)!} \sum_{a+b=h-1, a \neq h} \binom{h-1}{a} t^a (-u)^b, \quad i = 1, 2. \]

By Theorem 2 the first two terms of (7) tend to 0 uniformly in \( Q_w(s_0) \) as \( t \to +\infty \). Using (8), property (2), Theorem 2, and a method of Doetsch’s⁴ it can be shown that the third and fourth terms of (7) tend uniformly to 0 in \( Q_w(s_0) \) as \( t \to +\infty \). By Satz 4 of Doetsch⁵ and (2) the fifth term of (7) tends, as \( t \to +\infty \), to the existing integral

\[ \frac{1}{k(k!)} \int_0^\infty (s - s_0)^{k+2} e^{-(s-s_0)t} H_2(t) \, dt, \]

which converges uniformly in \( Q_w(s_0) \). Substituting from (1) into (9) we get (5). That the integral in (5) converges absolutely for \( R(s) > R(s_0) \) follows from (1) and the order relation (2). Assertions about (6) follow from preceding results, (2), and Theorem 2.

**Corollary 3.1.** If \( f_k \) exists, then the domain of \( f_k \) is a half-plane or the whole plane.

**Corollary 3.2.** Let \( f_k(s_0) \) exist and

\[ R(s) > R(s_0). \]

Then \( f_k(s) \) exists and the following representations of \( f_k(s) \) are valid:

\[ f_k(s) = \frac{(s - s_0)^{k+2}}{k!} \int_0^\infty e^{-(s-s_0)t} \left\{ \int_0^t e^{-su} dF(u) \right\} \cdot t^k \, dt, \]

\[ f_k(s) = \frac{(s - s_0)^{k+1}}{k!} \int_0^\infty e^{-(s-s_0)t} \left\{ \int_0^t e^{-su} dF(u) \right\} \cdot t^k \, dt, \]

where the integral in (10) converges absolutely for \( R(s) > R(s_0) \).

**Theorem 4.** If \( f_k(s_0) \) exists and if \( R(s) > R(s_0) \) or if \( s = s_0 \), then \( f_k(s) \) exists and the following representations of \( f_k(s) \) are valid:

\[ f_k(s) = \int_0^\infty \frac{(s - s_0)^{k+1}}{(k-1)!} e^{-(s-s_0)t} \left\{ \int_0^t e^{-su} dF(u) - f_k(s_0) \right\} \cdot t^{k-1} \, dt + f_k(s_0), \]

⁴ Ibid., p. 319.
⁵ Ibid., p. 34.
The integrals in (12) and (13) being uniformly convergent in every region $Q_w(s_0)$. The integral in (12) is absolutely convergent for $R(s) > R(s_0)$.

**Proof.** By (5) of Theorem 3 and the notation introduced in (1) we may write for $s$ in any region $Q_w(s_0)$, or for any $s$ such that $R(s) > R(s_0)$ or $s = s_0$,

\[
f_k(s) = \int_0^\infty \frac{(s - s_0)^k}{k!} e^{-(s-s_0)t} \left\{ \int_0^t e^{-u} dF(u) - f_k(s_0) \right\} dt + f_k(s_0),
\]

\[k = 1, 2, 3, \ldots \]

But since, by (1) and (2),

\[
\frac{H_2(t)}{k} = k \left\{ \int_0^t e^{-u} dF(u) - f_k(s_0) \right\} \left\{ \int_0^t e^{-u} dF(u) - f_k(s_0) \right\} dt
\]

as $t \to +\infty$, we can apply Satz 2 of Doetsch, Satz 4 of Doetsch, (1), and (2) to obtain (12) from (14), where the integral in (12) converges uniformly in $Q_w(s_0)$. The absolute convergence of the integral in (12) for $R(s) > R(s_0)$ follows from a consideration of (2). The uniform convergence in $Q_w(s_0)$ of the integral in (13) follows as in the proof of Theorem 3. The theorem is thus proved.

**Corollary 4.1.** Let $f_k(s_0)$ exist and $R(s) > R(s_0)$. Then $f_k(s)$ exists and the following representations of $f_k(s)$ are valid:

\[
f_k(s) = \frac{(s - s_0)^k}{(k-1)!} \int_0^\infty e^{-(s-s_0)t} \left\{ \int_0^t e^{-u} dF(u) \right\} dt,
\]

\[k = 1, 2, 3, \ldots \]

\[
f_k(s) = \frac{(s - s_0)^k}{(k-1)!} \int_0^\infty e^{-(s-s_0)t} \left\{ \int_0^t e^{-u} dF(u) \right\} dt,
\]

where the integral in (15) converges absolutely for $R(s) > R(s_0)$.

The proof of the next theorem follows by writing, for $k \geq 2$,
\[
\left[ \int_0^t e^{-su} dF(u) - f_k(s_0) \right] * t^{k-1} = (k - 1) \int_0^t \left\{ \left[ \int_0^u e^{-sv} dF(v) - f_k(s_0) \right] * u^{k-2} \right\} du,
\]
and then applying (12) of Theorem 4, (2), and Satz 2 of Doetsch:8

**Theorem 5.** Let \( f_k(s_0) \) exist and \( k \geq 2 \). Then \( f_k(s) \) exists for \( R(s) > R(s_0) \). If \( R(s) > R(s_0) \) or if \( s = s_0 \), the following representation of \( f_k(s) \) is valid:

\[
f_k(s) = \int_0^\infty \frac{(s - s_0)^k}{(k - 2)!} e^{-(s-s_0)t} \left\{ \left[ \int_0^t e^{-su} dF(u) - f_k(s_0) \right] * t^{k-2} \right\} dt + f_k(s_0).
\]
(17)

If \( R(s) > R(s_0) \), then also

\[
f_k(s) = \frac{(s - s_0)^k}{(k - 2)!} \int_0^\infty e^{-(s-s_0)t} \left\{ \left[ \int_0^t e^{-su} dF(u) \right] * t^{k-2} \right\} dt.
\]
(18)

4. Function theoretic properties of the transform. In this section we use the notation \( L^{(k)}(F) = f_k \).

**Theorem 6.** Let \( F_1 \) and \( F_2 \) be two functions from the class \( \{ F \} \) such that \( F_1(0) = F_2(0) \) and such that

\( L^{(k)}(F_1) = L^{(k)}(F_2) = f_k \).

Then \( F_1(t) = F_2(t) \) almost everywhere for \( t \geq 0 \).

**Proof.** Since \( f_k \) exists, there is a real \( s_0 > 0 \) such that for all \( s \) where \( R(s) > s_0 \) \( f_k(s) \) is given by (15). Defining

\[
G_i(t) = \int_0^t e^{-su} dF_i(u), \quad i = 1, 2,
\]
(19)

we easily get from (15), for all \( s \) where \( R(s) > s_0 \),

\[
\int_0^\infty e^{-(s-s_0)t} \left\{ \left[ G_1(t) - G_2(t) \right] * t^{k-1} \right\} dt = 0.
\]
(20)

By Theorem 5c of Widder9 \( G_1(t) - G_2(t) \) is of bounded variation on every closed interval \([0, T]\), \( T \geq 0 \) and by Satz 4 of Doetsch10

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8 Ibid., p. 90.
[G_1(t) - G_2(t)]^{*k-1} is a J-Function "in the sense of Doetsch."^{11} By Satz 4, of Doetsch^{12} Satz 12 of Doetsch,^{13} and Theorem 11.52 of Titchmarsh^{14} we get that G_1(t) = G_2(t) almost everywhere for t \geq 0. Referring to (19) and (20) we may now write

\begin{equation}
\int_0^t e^{-s_0 v} [F_1(v) - F_2(v)] = 0
\end{equation}

almost everywhere for t \geq 0. Integrating (21) by parts and using the hypothesis F_1(0) = F_2(0) we get

\begin{equation}
e^{-s_0 t} [F_1(t) - F_2(t)] + s_0 \int_0^t e^{-s_0 v} [F_1(v) - F_2(v)] dv = 0
\end{equation}

almost everywhere for t \geq 0. Theorem 6 now follows easily from (22).

Theorem 6 now yields the following theorem:

**Theorem 7.** Let F_1 and F_2 be two functions from the class \{F\} such that

\[ L^{(k)}(F_1) = L^{(k)}(F_2) = f_k. \]

Then F_1(t) = F_2(t) + [F_1(0) - F_2(0)] almost everywhere for t \geq 0.

**Theorem 8.** Let f_k(s_0) exist. Then f_k is analytic in the region defined by R(s) > R(s_0).

**Proof.** The theorem follows from the fact that f_k(s) is representable by a Laplace or Laplace-Stieltjes integral if R(s) > R(s_0).

**Theorem 9.** f_k is analytic in the half-plane, or the whole plane, of its existence.

**Proof.** The theorem follows from Corollary 3.1 and Theorem 8.

The following theorem is easily proved:

**Theorem 10.** M_k(s, t) is an integral function of s.

**Theorem 11.** Let f_k(s_0) exist. Then

f_k(s_0) \rightarrow f_k(s_0)

if \{s_{a}\} is any sequence in any region Q_w(s_0) such that \{s_{a}\} \rightarrow s_0.

**Proof.** The theorem follows from Theorem 3 and Theorem 10.

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11 Ibid., p. 29.
12 Ibid., p. 74.
13 Ibid., p. 131.
THEOREM 12. If \( f_k(s_0) \) exists, then for any \( c > R(s_0) \) and \( t \geq 0 \)
\[
\left[ \int_0^t e^{-su}dF(u) \right] * t^{k-1} = \lim_{T \to +\infty} \frac{(k-1)!}{2\pi i} \int_{c-\pi i}^{c+\pi i} \frac{f_k(s)}{(s-s_0)^{k+1}} e^{st} ds.
\]

PROOF. We assume \( f_k(s_0) \) exists, \( c > R(s_0) \), and \( t \geq 0 \). Let
\[
H(t) = \left[ \int_0^t e^{-su}dF(u) \right] * t^{k-1}.
\]

By (15) of Corollary 4.1 and (23) we have
\[
f_k(s) = \frac{(s-s_0)^{k+1}}{(k-1)!} \int_0^\infty e^{-(s-s_0)t} H(t) dt,
\]
where the integral converges absolutely for \( R(s) > R(s_0) \). By reasoning similar to that in the proof of Theorem 6 we can assert that \( H \) is a \( \mathcal{J} \)-Function "in the sense of Doetsch," and that \( H \) is continuous and of bounded variation on every finite, closed interval \([0, T]\), \( T \geq 0 \). Hence we can apply Satz 3 of Doetsch to
\[
L(H) = \frac{(k-1)!f_k(s)}{(s-s_0)^{k+1}}
\]
to obtain
\[
\left[ \int_0^t e^{-su}dF(u) \right] * t^{k-1} = \lim_{T \to +\infty} \frac{(k-1)!}{2\pi i} \int_{c-\pi i}^{c+\pi i} \frac{f_k(s)}{(s-s_0)^{k+1}} e^{st} ds.
\]

18 Ibid., p. 212.