A GENERALIZED LAPLACE-STIELTJES TRANSFORMATION

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1. **Introduction.** Gustav Doetsch[1] has recently introduced a generalized Laplace transform of order k, $k \geqq 0$. It is the object of this paper to introduce a generalized Laplace-Stieltjes transform of order k.

We assume in what follows that k is positive and integral. We also assume that F denotes a complex-valued function of a real variable, which is defined, except perhaps on a set of isolated points, and is of bounded variation on every closed, finite interval $[0, T]$, $T \geqq 0$. By $\{F\}$ we mean the class of all functions F defined above. All integrals in this paper are to be interpreted as Riemann or Riemann-Stieltjes integrals. The symbol (*) will denote a convolution.

2. **Definition of the transform.** Let s be complex. Let $M_k(s, t)$ be the Cesàro mean of order k of

$$\int_0^t e^{-su} dF(u),$$

i.e.,

$$M_k(s, t) = \frac{k}{t^k}\left\{\left[\int_0^t e^{-su} dF(u)\right] * t^{k-1}\right\}.$$

If for some s and some k

$$\lim_{t \to +\infty} M_k(s, t)$$

exists, then this limit defines a value $f_k(s)$ of a function f_k:

$$f_k(s) = \lim_{t \to +\infty} M_k(s, t).$$

We call f_k the generalized Laplace-Stieltjes transform of order k of F.

3. **Existence and integral representations of the transform.** We assume in what follows that s_0 is a fixed complex number. From the definition of $f_k(s_0)$ and the consistency of Cesàro summability we can assert the following:

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[1] Gustav Doetsch, *Handbuch der Laplace-Transformation*, Basel, Verlag Birkhauser, pp. 311–352.

THEOREM 1. *If $f_k(s_0)$ exists and N is a nonnegative integer, then, as $t\to+\infty$,*

$$\left[\int_0^t e^{-s_0 u}dF(u) - f_k(s_0)\right] * t^{k+N-1} = o(t^{k+N}).$$

We now define

$$H_i(t) = k\left[\int_0^t e^{-s_0 u}dF(u) - f_k(s_0)\right] * t^{k-2+i}, \qquad i = 1, 2. \tag{1}$$

Then, by Theorem 1, if $f_k(s_0)$ exists,

$$H_i(t) = o(t^{k-1+i}), \qquad i = 1, 2, \tag{2}$$

as $t\to+\infty$. We also define

$$G(t) = \int_0^t e^{-s_0 u}dF(u), \tag{3}$$

so that we may now write (1) as

$$H_i(t) = k[G(t) - f_k(s_0)] * t^{k-2+i}, \qquad i = 1, 2. \tag{4}$$

In what follows we define an angular region $Q_w(s_0)$ to be the totality of all s such that $|\arg(s-s_0)| \leqq w$, where w is a fixed number satisfying $0\leqq w<\pi/2$.

Using the facts that $|s-s_0| \leqq R(s-s_0)/\cos w$ for every s in $Q_w(s_0) - \{s_0\}$ and that for every $\epsilon_i>0$ there exists, by (2), a T_i such that $|H_i(t)| < \epsilon_i t^{k-1+i} (\cos^{h-1+i} w)/(h-1+i)!$ whenever $t>T_i$, we can establish the following theorem:

THEOREM 2. *Let $f_k(s_0)$ exist. Let $Q_w(s_0)$ be any fixed angular region. Then for any $\epsilon_i>0$ and nonnegative integer h a T_i exists such that uniformly in $Q_w(s_0)$*

$$\left| (s - s_0)^{h-1+i}e^{-(s-s_0)t}H_i(t) \right| < \epsilon_i t^{k-h}$$

whenever $t>T_i$, $i=1, 2$.

THEOREM 3. *If $f_k(s_0)$ exists and $R(s)>R(s_0)$, then $f_k(s)$ exists. Furthermore, $M_k(s, t)$ converges uniformly to $f_k(s)$ in every region $Q_w(s_0)$. If $R(s)>R(s_0)$ or if $s=s_0$, the following representations of $f_k(s)$ are valid:*

$$\begin{aligned} f_k(s) = &\int_0^\infty \frac{(s-s_0)^{k+2}}{k!} e^{-(s-s_0)t}\left\{\left[\int_0^t e^{-s_0 u}dF(u) - f_k(s_0)\right] * t^k\right\} dt \\ &+ f_k(s_0), \end{aligned} \tag{5}$$

$$f_k(s) = \int_0^\infty \frac{(s-s_0)^{k+1}}{k!} e^{-(s-s_0)t} d\left\{\left[\int_0^t e^{-s_0u}dF(u) - f_k(s_0)\right] * t^k\right\} + f_k(s_0). \tag{6}$$

The integrals in (5) *and* (6) *are uniformly convergent in every region* $Q_w(s_0)$. *The integral in* (5) *is absolutely convergent for* $R(s) > R(s_0)$.

PROOF. Assertions of the theorem for the case $s = s_0$ can be easily verified. We therefore assume $s \neq s_0$. Reflection now reveals that it is immaterial whether we consider s to be such that $R(s) > R(s_0)$ or whether we consider it to be such that it lies in some region $Q_w(s_0) - \{s_0\}$. We therefore further assume that s is in some region $Q_w(s_0) - \{s_0\}$.

By Theorem 6b of Widder[2] and (3) we may write, since $R(s) > R(s_0)$,

$$\int_0^t e^{-su}dF(u) = \int_0^t e^{-(s-s_0)u}dG(u).$$

From this it follows that

$$M_k(s, t) = \frac{k}{t^k}\int_0^t \left\{\int_0^u e^{-(s-s_0)v}dG(v)\right\}(t-u)^{k-1}du,$$

or, since the Stieltjes integral with respect to a constant is 0,

$$M_k(s, t) = \frac{k}{t^k}\int_0^t \left\{\int_0^u e^{-(s-s_0)v}d[G(v) - f_k(s_0)]\right\}(t-u)^{k-1}du.$$

After integrating by parts, using a method of Doetsch's[3] and referring to (4) we get

$$\begin{aligned} M_k(s, t) = {} & \frac{e^{-(s-s_0)t}}{t^k} H_1(t) + \frac{(s-s_0)e^{-(s-s_0)t}}{kt^k} H_2(t) \\ & + \frac{1}{t^k}\int_0^t I_1(u, t)e^{-(s-s_0)u}H_1(u)du \\ & + \frac{1}{kt^k}\int_0^t I_2(u, t)e^{-(s-s_0)u}H_2(u)du \\ & + \frac{1}{k(k!)}\int_0^t (s-s_0)^{k+2}e^{-(s-s_0)u}H_2(u)du + f_k(s_0), \end{aligned} \tag{7}$$

where

[2] David Vernon Widder, *The Laplace transform*, Princeton, Princeton University Press, 1946, p. 12.

[3] Doetsch, op. cit., pp. 316–317.

$$
(8) \qquad \begin{aligned} I_i(u, t) = & \sum_{h=1}^{k-1+i} \binom{k-1+i}{h} \frac{(s-s_0)^{h-1+i}}{(h-1)!} \\ & \cdot \sum_{a+b=h-1,\, a\neq k} \binom{h-1}{a} t^a(-u)^b, \qquad i = 1, 2. \end{aligned}
$$

By Theorem 2 the first two terms of (7) tend to 0 uniformly in $Q_w(s_0)$ as $t \to +\infty$. Using (8), property (2), Theorem 2, and a method of Doetsch's[4] it can be shown that the third and fourth terms of (7) tend uniformly to 0 in $Q_w(s_0)$ as $t \to +\infty$. By Satz 4 of Doetsch[5] and (2) the fifth term of (7) tends, as $t \to +\infty$, to the existing integral

$$
(9) \qquad \frac{1}{k(k!)} \int_0^\infty (s - s_0)^{k+2} e^{-(s-s_0)t} H_2(t)\,dt,
$$

which converges uniformly in $Q_w(s_0)$. Substituting from (1) into (9) we get (5). That the integral in (5) converges absolutely for $R(s) > R(s_0)$ follows from (1) and the order relation (2). Assertions about (6) follow from preceding results, (2), and Theorem 2.

COROLLARY 3.1. *If f_k exists, then the domain of f_k is a half-plane or the whole plane.*

COROLLARY 3.2. *Let $f_k(s_0)$ exist and*

$$
R(s) > R(s_0).
$$

Then $f_k(s)$ exists and the following representations of $f_k(s)$ are valid:

$$
(10) \qquad f_k(s) = \frac{(s-s_0)^{k+2}}{k!} \int_0^\infty e^{-(s-s_0)t} \left\{ \left[\int_0^t e^{-s_0 u} dF(u) \right] * t^k \right\} dt,
$$

$$
(11) \qquad f_k(s) = \frac{(s-s_0)^{k+1}}{k!} \int_0^\infty e^{-(s-s_0)t} d \left\{ \left[\int_0^t e^{-s_0 u} dF(u) \right] * t^k \right\},
$$

where the integral in (10) *converges absolutely for $R(s) > R(s_0)$.*

THEOREM 4. *If $f_k(s_0)$ exists and if $R(s) > R(s_0)$ or if $s = s_0$, then $f_k(s)$ exists and the following representations of $f_k(s)$ are valid:*

$$
(12) \qquad \begin{aligned} f_k(s) = & \int_0^\infty \frac{(s-s_0)^{k+1}}{(k-1)!} e^{-(s-s_0)t} \\ & \cdot \left\{ \left[\int_0^t e^{-s_0 u} dF(u) - f_k(s_0) \right] * t^{k-1} \right\} dt + f_k(s_0), \end{aligned}
$$

[4] Ibid., p. 319.
[5] Ibid., p. 34.

$$
(13)\quad \begin{aligned} f_k(s) = \int_0^{\infty} &\frac{(s-s_0)^k}{(k-1)!} e^{-(s-s_0)t} \\ &\cdot d\left\{\left[\int_0^t e^{-s_0u}dF(u) - f_k(s_0)\right] * t^{k-1}\right\} + f_k(s_0), \end{aligned}
$$

the integrals in (12) *and* (13) *being uniformly convergent in every region* $Q_w(s_0)$. *The integral in* (12) *is absolutely convergent for* $R(s) > R(s_0)$.

PROOF. By (5) of Theorem 3 and the notation introduced in (1) we may write for s in any region $Q_w(s_0)$, or for any s such that $R(s) > R(s_0)$ or $s = s_0$,

$$
(14)\quad f_k(s) = \int_0^{\infty} \frac{(s-s_0)^{k+2}}{k!} e^{-(s-s_0)t} \left\{\frac{H_2(t)}{k}\right\} dt + f_k(s_0).
$$

But since, by (1) and (2),

$$
\begin{aligned} \frac{H_2(t)}{k} &= k\left\{\left[\int_0^t e^{-s_0u}dF(u) - f_k(s_0)\right] * t^{k-1}\right\} * 1 \\ &= \int_0^t H_1(u)du = o(t^{k+1}) \end{aligned}
$$

as $t \to +\infty$, we can apply Satz 2 of Doetsch,[6] Satz 4 of Doetsch,[7] (1), and (2) to obtain (12) from (14), where the integral in (12) converges uniformly in $Q_w(s_0)$. The absolute convergence of the integral in (12) for $R(s) > R(s_0)$ follows from a consideration of (2). The uniform convergence in $Q_w(s_0)$ of the integral in (13) follows as in the proof of Theorem 3. The theorem is thus proved.

COROLLARY 4.1. *Let* $f_k(s_0)$ *exist and* $R(s) > R(s_0)$. *Then* $f_k(s)$ *exists and the following representations of* $f_k(s)$ *are valid*:

$$
(15)\quad f_k(s) = \frac{(s-s_0)^{k+1}}{(k-1)!} \int_0^{\infty} e^{-(s-s_0)t} \left\{\left[\int_0^t e^{-s_0u}dF(u)\right] * t^{k-1}\right\} dt,
$$

$$
(16)\quad f_k(s) = \frac{(s-s_0)^{k}}{(k-1)!} \int_0^{\infty} e^{-(s-s_0)t} d\left\{\left[\int_0^t e^{-s_0u}dF(u)\right] * t^{k-1}\right\},
$$

where the integral in (15) *converges absolutely for* $R(s) > R(s_0)$.

The proof of the next theorem follows by writing, for $k \geqq 2$,

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[6] Ibid., p. 90.

[7] Ibid., p. 34.

$$\left[\int_0^t e^{-s_0 u} dF(u) - f_k(s_0)\right] * t^{k-1}$$
$$= (k-1) \int_0^t \left\{\left[\int_0^u e^{-s_0 v} dF(v) - f_k(s_0)\right] * u^{k-2}\right\} du,$$

and then applying (12) of Theorem 4, (2), and Satz 2 of Doetsch:[8]

THEOREM 5. *Let $f_k(s_0)$ exist and $k \geqq 2$. Then $f_k(s)$ exists for $R(s) > R(s_0)$. If $R(s) > R(s_0)$ or if $s = s_0$, the following representation of $f_k(s)$ is valid:*

$$f_k(s) = \int_0^\infty \frac{(s-s_0)^k}{(k-2)!} e^{-(s-s_0)t} \left\{\left[\int_0^t e^{-s_0 u} dF(u) - f_k(s_0)\right] * t^{k-2}\right\} dt + f_k(s_0). \tag{17}$$

If $R(s) > R(s_0)$, then also

$$f_k(s) = \frac{(s-s_0)^k}{(k-2)!} \int_0^\infty e^{-(s-s_0)t} \left\{\left[\int_0^t e^{-s_0 u} dF(u)\right] * t^{k-2}\right\} dt. \tag{18}$$

4. Function theoretic properties of the transform. In this section we use the notation $L^{(k)}(F) = f_k$.

THEOREM 6. *Let F_1 and F_2 be two functions from the class $\{F\}$ such that $F_1(0) = F_2(0)$ and such that*

$$L^{(k)}(F_1) = L^{(k)}(F_2) = f_k.$$

Then $F_1(t) = F_2(t)$ almost everywhere for $t \geqq 0$.

PROOF. Since f_k exists, there is a real $s_0 > 0$ such that for all s where $R(s) > s_0$ $f_k(s)$ is given by (15). Defining

$$G_i(t) = \int_0^t e^{-s_0 u} dF_i(u), \qquad i = 1, 2, \tag{19}$$

we easily get from (15), for all s where $R(s) > s_0$,

$$\int_0^\infty e^{-(s-s_0)t} \{ [G_1(t) - G_2(t)] * t^{k-1} \} dt = 0. \tag{20}$$

By Theorem 5c of Widder[9] $G_1(t) - G_2(t)$ is of bounded variation on every closed interval $[0, T]$, $T \geqq 0$ and by Satz 4 of Doetsch[10]

[8] Ibid., p. 90.
[9] Op. cit., p. 9.
[10] Op. cit., p. 113.

$[G_1(t)-G_2(t)]*t^{k-1}$ is a J-Function "in the sense of Doetsch."[11] By Satz 4, of Doetsch[12] Satz 12 of Doetsch,[13] and Theorem 11.52 of Titchmarsh[14] we get that $G_1(t)=G_2(t)$ almost everywhere for $t\geqq 0$. Referring to (19) and (20) we may now write

$$\int_0^t e^{-s_0 v}d[F_1(v) - F_2(v)] = 0 \tag{21}$$

almost everywhere for $t\geqq 0$. Integrating (21) by parts and using the hypothesis $F_1(0)=F_2(0)$ we get

$$e^{-s_0 t}[F_1(t) - F_2(t)] + s_0\int_0^t e^{-s_0 v}[F_1(v) - F_2(v)]dv = 0 \tag{22}$$

almost everywhere for $t\geqq 0$. Theorem 6 now follows easily from (22).

Theorem 6 now yields the following theorem:

THEOREM 7. *Let F_1 and F_2 be two functions from the class $\{F\}$ such that*

$$L^{(k)}(F_1) = L^{(k)}(F_2) = f_k.$$

Then $F_1(t)=F_2(t)+[F_1(0)-F_2(0)]$ almost everywhere for $t\geqq 0$.

THEOREM 8. *Let $f_k(s_0)$ exist. Then f_k is analytic in the region defined by $R(s)>R(s_0)$.*

PROOF. The theorem follows from the fact that $f_k(s)$ is representable by a Laplace or Laplace-Stieltjes integral if $R(s)>R(s_0)$.

THEOREM 9. *f_k is analytic in the half-plane, or the whole plane, of its existence.*

PROOF. The theorem follows from Corollary 3.1 and Theorem 8.

The following theorem is easily proved:

THEOREM 10. *$M_k(s, t)$ is an integral function of s.*

THEOREM 11. *Let $f_k(s_0)$ exist. Then*

$$f_k(s_\alpha) \to f_k(s_0)$$

if $\{s_\alpha\}$ is any sequence in any region $Q_w(s_0)$ such that $\{s_\alpha\}\to s_0$.

PROOF. The theorem follows from Theorem 3 and Theorem 10.

[11] Ibid., p. 29.

[12] Ibid., p. 74.

[13] Ibid., p. 131.

[14] E. C. Titchmarsh, *The theory of functions*, 2d ed. London, Oxford University Press, 1939, p. 360.

THEOREM 12. *If $f_k(s_0)$ exists, then for any $c>R(s_0)$ and $t\geqq 0$*

$$\left[\int_0^t e^{-s_0u}dF(u)\right]*t^{k-1} = \operatorname*{Lim}_{T\to+\infty}\frac{(k-1)!}{2\pi i}\int_{c-iT}^{c+iT}\frac{f_k(s)}{(s-s_0)^{k+1}}e^{st}ds.$$

PROOF. We assume $f_k(s_0)$ exists, $c>R(s_0)$, and $t\geqq 0$. Let

$$H(t) = \left[\int_0^t e^{-s_0u}dF(u)\right]*t^{k-1}. \tag{23}$$

By (15) of Corollary 4.1 and (23) we have

$$f_k(s) = \frac{(s-s_0)^{k+1}}{(k-1)!}\int_0^\infty e^{-(s-s_0)t}H(t)dt,$$

where the integral converges absolutely for $R(s)>R(s_0)$. By reasoning similar to that in the proof of Theorem 6 we can assert that H is a J-Function "in the sense of Doetsch,"[15] and that H is continuous and of bounded variation on every finite, closed interval $[0, T]$, $T\geqq 0$. Hence we can apply Satz 3 of Doetsch[16] to

$$L(H) = \frac{(k-1)!f_k(s)}{(s-s_0)^{k+1}}$$

to obtain

$$\left[\int_0^t e^{-s_0u}dF(u)\right]*t^{k-1} = \operatorname*{Lim}_{T\to+\infty}\frac{(k-1)!}{2\pi i}\int_{c-iT}^{c+iT}\frac{f_k(s)}{(s-s_0)^{k+1}}e^{st}ds.$$

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[15] Op. cit., p. 29.

[16] Ibid., p. 212.