ON THE RELATIVE DENSITY OF SETS OF INTEGERS[1]

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In [1], U. T. Medvedev introduced the concept of relative density of sets of non-negative integers. A set α_2 is *at least as dense as* a set α_1 when there exists a general recursive function $g(x)$ such that, for every n, the number of elements of α_1 not greater than n does not exceed the number of elements of α_2 not greater than $g(n)$. We give here an equivalent definition and prove a theorem which has a result of Medvedev and a result of the author as corollaries.

We assume familiarity with [3], especially §§8, 16 and 17.

1. If $h_i(x)$ produces the principal sequence of α_i, the number of elements of α_i not greater than n is $\mu y[n<h_i(y)]$. So α_2 is at least as dense as α_1 when there exists a general recursive function $g(x)$ such that, for all n, $\mu y[n<h_1(y)]\leqq\mu y[g(n)<h_2(y)]$. We show that this is equivalent to $h_2(x)\leqq g(h_1(x))$ for all x.

Suppose for all n, $\mu y[n<h_1(y)]\leqq\mu y[g(n)<h_2(y)]$. Then if for some x, $g(h_1(x))<h_2(x)$ we have the contradiction

$$x \geqq \mu y[g(h_1(x)) < h_2(y)] \geqq \mu y[h_1(x) < h_1(y)] = x + 1.$$

On the other hand, suppose for a general recursive function $g(x)$, $h_2(x)\leqq g(h_1(x))$. We may assume $g(x)$ to be monotone increasing without loss of generality. Then for any n, $h_1(\mu y[n<h_1(y)]-1)\leqq n$. Hence

$$h_2(\mu y[n < h_1(y)] - 1) \leqq g(h_1(\mu y[n < h_1(y)] - 1)) \leqq g(n).$$

So

$$\begin{aligned}\mu y[g(n) < h_2(y)] &> \mu y[n < h_1(y)] - 1\\ &\geqq \mu y[n < h_1(y)].\end{aligned}$$

We can say, then that α_2 is at least as dense as α_1 when there exists a general recursive function which maps α_1 onto a set whose principal sequence bounds that of α_2. In particular, if $\alpha\in B$, then α is at least as dense as ϵ; this class constitutes the highest degree of density. The relation "is at least as dense as" is clearly reflexive and transitive.

We mention a fact observed by Medvedev: if $\alpha_1\in V-B$, then the set $\alpha_2=\alpha_1-\{a\}$, formed by removing the element a from α_1, is less

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dense than α_1. For when x is greater than $\mu y[h_1(y)=a]$, $h_1(x)<h_1(x+1)=h_2(x)$. So α_1 is at least as dense as α_2. However, if α_2 were at least as dense as α_1, there would exist a monotone increasing general recursive function $g(x)$ such that $h_2(x)=h_1(x+1)\leqq g(h_1(x))\leqq g^{x+1}(h_1(0))$. Thus $\alpha_1 \in B$, contrary to hypothesis.

2. Theorem. *B^* is the class of automorphisms of V which preserve or increase density.*

Proof. We consider the automorphism $\alpha \rightarrow p(\alpha)$, and show that for all sets α, $p(\alpha)$ is at least as dense as α, if and only if there exists a general recursive function $f(x)$ such that $p(x)\leqq f(x)$ for all x.

First, suppose $p(x)\leqq f(x)$. Then if $\alpha_2=p(\alpha_1)$,

$$\begin{aligned} h_2(x) &\leqq \max\,[p(h_1(0)), p(h_1(1)), \cdots, p(h_1(x))] \\ &\leqq \max\,[f(0), f(1), \cdots, f(h_1(x))]. \end{aligned}$$

So $g(x)=\max\,[f(0), f(1), \cdots, f(x)]$ is a general recursive function for which $h_2(x)\leqq g(h_1(x))$, and $\alpha_2=p(\alpha_1)$ is at least as dense as α_1.

Conversely, suppose that $p(\alpha)$ is at least as dense as α for every set α. Let α_2 be the set of maximal values of $p(x)$, and α_1 the set of all y such that $p(y)\in\alpha_2$. Then $\alpha_2=p(\alpha_1)$; in fact $h_2(x)=p(h_1(x))$ and $h_1(0)=0$. Now $h_1(x+1)\leqq h_2(x)+1$, for at least one of the $h_2(x)+2$ values of $p(y)$ for $0\leqq y\leqq h_2(x)+1$ must exceed $h_2(x)$. $h_1(x+1)$ produces the principal sequence of $\alpha_1-\{0\}$, so we have shown that $\alpha_1-\{0\}$ is at least as dense as α_2. But α_2 is at least as dense as α_1 by hypothesis, and α_1 is at least as dense as $\alpha_1-\{0\}$. Hence all three sets are of equal density. Since α_1 and $\alpha_1-\{0\}$ can be of equal density only when $\alpha_1\in B$, we have also $\alpha_2\in B$ and by Theorem 22 of [**3**], $p(x)\leqq f(x)$ for some general recursive function $f(x)$.

Corollary [**3**, Theorem 23, Corollary]. *B is closed under automorphisms in B^*.*

Corollary. [1] *Density is preserved under automorphisms in E^*.*

For density is preserved by just those automorphisms which are in B^* and whose inverses are also in B^*.

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