COMMENT ON "THE DISTANCE TO THE ORIGIN OF A CERTAIN POINT SET IN $E^n"$

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In a recent paper [1] Karush and Wolfsohn have considered (in slightly different notation) the problem of finding

$$M = \min \sum_{i=0}^{m} a_i^2.$$ 

Subject to the conditions

$$\sum_{i=0}^{m} a_i = 1, \quad \sum_{i=0}^{m} i^k a_i = 0 \quad \text{for } k = 1, 2, \ldots, r$$

where $r$ is a fixed integer between 0 and $m-1$ inclusive. After somewhat lengthy calculations they obtain the result

$$M = 1 - \frac{(m!)^2}{(m + r + 1)!(m - r - 1)!}.$$ 

The purpose of this note is to obtain the result by simpler, more geometrical considerations; in particular we employ the theory, treated for example in [2, §72 et seq.], of polynomials orthogonal over a finite set.

Let $P_{k,m}(x) \equiv P_k(x)$ be the orthogonal polynomials of degrees $k = 0, 1, \ldots, m$ on the set $x = 0, 1, \ldots, m$ having unit constant terms; thus $\sum_{k=0}^{m} P_k(x) P_j(x) = 0$ if $k \neq j$, or, in vector notation, $(P_k, P_j) = 0$ if $k \neq j$, where $P_k$ denotes the vector with components $P_k(x), x = 0, 1, \ldots, m$.

Let $Q_k$ be the corresponding unit vectors, $Q_k = P_k/\|P_k\|$. The squares of the normalizing constants can be written (cf. [2, p. 268, eq. (9)])

$$\frac{1}{\|P_k\|^2} = \frac{2k + 1}{m + k + 1} \cdot \frac{(m!)^2}{(m + k)!(m - k)!}.$$ 

Since $P_k(0) = 1$ we have

$$Q_k(0)^2 = \frac{2k + 1}{m + k + 1} \cdot \frac{(m!)^2}{(m + k)!(m - k)!}.$$ 

Let $a = (a_0, a_1, \ldots, a_m)$ be a vector satisfying (1). The inner product of $a$ with any polynomial $Q$ of degree $\leq r$ is clearly given by

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\[(a, Q) = Q(0),\]
and in particular, for the orthonormal polynomials \(Q_k,\)
\[(a, Q_k) = Q_k(0), \quad k = 0, 1, \ldots, r.\]

Now the vectors \(Q_k, k = 0, 1, \ldots, m,\) span \(E^{m+1}\) and are orthonormal. Hence any vector \(a \in E^{m+1}\) may be written
\[a = \sum_{k=0}^{m} \alpha_k Q_k, \quad \alpha_k = (\alpha, Q_k)\]
with
\[\|a\|^2 = \sum_{i=0}^{m} a_i^2 = \sum_{k=0}^{m} \alpha_k^2.\]

Then the solution to the minimum problem is obtained by choosing
\[\alpha_k = Q_k(0) \quad \text{for } k = 0, 1, \ldots, r,\]
\[\alpha_k = 0 \quad \text{for } k = r + 1, \ldots, m.\]

Hence
\[M = \sum_{k=0}^{r} Q_k(0)^2\]
\[= \sum_{k=0}^{r} \frac{2k + 1}{m + k + 1} \frac{(m!)^2}{(m + k)! (m - k)!}\]
\[= \sum_{k=0}^{r} \frac{(m + k + 1)(m - k)}{m + k + 1} \frac{(m!)^2}{(m + k)! (m - k)!}\]
\[= \sum_{k=0}^{r} \frac{(m!)^2}{(m + k)! (m - k)!} - \sum_{k=0}^{r} \frac{(m!)^2}{(m + k + 1)! (m - k - 1)!}\]
\[= \left( \sum_{k=0}^{r} - \sum_{k=0}^{r+1} \right) \frac{(m!)^2}{(m + k)! (m - k)!}\]
\[= 1 - \frac{(m!)^2}{(m + r + 1)! (m - r - 1)!},\]
in agreement with (2).

References

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