ON THE COVERING OF \( E_n \) BY SPHERES

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1. **Statement of results.** Let \( S_n \) denote the set of (closed) \( n \) -spheres with radii of length \( (n^{1/2})/2 \) and centers on the lattice points of a rectangular Cartesian coordinate system in \( E_n \) (Euclidean \( n \)-space). Since \( (n^{1/2})/2 \) is half the largest diagonal of the unit \( n \)-cube, every point of \( E_n \) falls on or within some of the spheres of \( S_n \). For \( n = 1, 2, 3 \), if an \( n \)-sphere is removed from \( S_n \), then certain points of \( E_n \) are not covered by the remaining spheres. However, for \( n > 3 \), proper subsets of \( S_n \) cover \( E_n \) completely.

Let \( \lfloor x \rfloor \) denote the greatest integer less than or equal to \( x \). We prove the following theorem:

**Theorem.** Each point of \( E_n \) is on or within some \( n \)-sphere with radius of length \( (n^{1/2})/2 \) and center at a lattice point \((y_1, \ldots, y_n)\) for which

\[
\sum_{i=1}^{n} y_i \equiv 0 \mod (\lfloor n/4 \rfloor + 1).
\]

2. **A lemma.** Let

1) \((x_1, x_2, x_3, x_4)\) be any point of \( E_4 \),
2) \(\delta_i = x_i - (\lfloor x_i \rfloor + 1/2) \) \((i = 1, 2, 3, 4)\),
3) \((i_1, i_2, i_3, i_4)\) be a rearrangement of \((1, 2, 3, 4)\) such that

\[
|\delta_{i_1}| \leq |\delta_{i_2}| \leq |\delta_{i_3}| \leq |\delta_{i_4}|.
\]

4) \((y_1', y_2', y_3', y_4')\) and \((y_1'', y_2'', y_3'', y_4'')\) be points in \( E_4 \) defined as follows:

(i) \(y_1'' = [x_{i_1}]\) and \(y_1'' = [x_{i_1}] + 1\),

(ii) for \(j = 2, 3, 4\)

\[
y_{ij}' = y_{ij}'' = \begin{cases} [x_{ij}] & \text{if } \delta_{ij} \leq 0, \\
[x_{ij}] + 1 & \text{if } \delta_{ij} > 0.
\end{cases}
\]

Then

\[
\sum_{i=1}^{4} (x_i - y_i')^2 \leq 1
\]

and

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Proof. Let \( k \) range over the set \( \{y'_i, y''_i\} \) and \( j \) over the set \( \{2, 3, 4\} \). For each \( j \) such that \( \delta_{ij} \leq 0 \)

\[
(x_{ij} - k) = \left( [x_{ij}] + \frac{1}{2} - |\delta_{ij}| - [x_{ij}] \right) = \left( \frac{1}{2} - |\delta_{ij}| \right).
\]

For each \( j \) such that \( \delta_{ij} > 0 \)

\[
(x_{ij} - k) = \left( [x_{ij}] + \frac{1}{2} - |\delta_{ij}| - ([x_{ij}] + 1) \right)
= \left( |\delta_{ij}| - \frac{1}{2} \right).
\]

Therefore

\[
\sum_{i=1}^{4} (x_{ij} - k)^2 = \sum_{i=1}^{4} \left( \frac{1}{2} - |\delta_{ij}| \right)^2.
\]

Proof of (2): (i) Suppose \( \delta_{i1} \leq 0 \). Then (4) holds if we replace \( k \) by \( y'_i \) and all \( j \) by 1. Since \( |\delta_{ij}| \leq 1/2 \)

\[
\sum_{i=1}^{4} (x_{ij} - y'_i)^2 = \sum_{i=1}^{4} \left( \frac{1}{2} - |\delta_{ij}| \right)^2 \leq 1.
\]

(ii) Suppose \( \delta_{i1} > 0 \). Then

\[
(x_{ij} - y'_i) = \left( [x_{i1}] + \frac{1}{2} + |\delta_{i1}| - [x_{i1}] \right) = \left( \frac{1}{2} + |\delta_{i1}| \right).
\]

Therefore

\[
\sum_{i=1}^{4} (x_{ij} - y''_i)^2 = \left( \frac{1}{2} + |\delta_{i1}| \right)^2 + \sum_{i=2}^{4} \left( \frac{1}{2} - |\delta_{ij}| \right)^2 
\leq \left( \frac{1}{2} + |\delta_{i1}| \right)^2 + 3 \left( \frac{1}{2} - |\delta_{i1}| \right)^2 
= 1 - 2 |\delta_{i1}| + 4 |\delta_{i1}|^2 
\leq 1
\]

since \( |\delta_{i1}| \leq 1/2 \) implies \( 4 |\delta_{i1}|^2 \leq 2 |\delta_{i1}| \).

Proof of (3): (i) Suppose \( \delta_{i1} > 0 \). Then (5) holds if we replace \( k \) by \( y''_i \) and all \( j \) by 1. Therefore (7) holds if we replace \( y'_i \) by \( y''_i \).

(ii) Suppose \( \delta_{i1} \leq 0 \). Then
\[(x_{i_1} - y_{i_1}^{''}) = \left(\left[x_{i_1}\right] + \frac{1}{2} - \left|\delta_{i_1}\right| - \left([x_{i_1}] + 1\right)\right)\]
\[(10) = \left(-\frac{1}{2} - \left|\delta_{i_1}\right|\right).\]

Therefore (9) holds if we replace \(y_{i_1}^{'}\) by \(y_{i_1}^{''}\).

It is an immediate consequence of the lemma that the theorem is true for \(n = 4\):

**Corollary.** Each point of \(E_4\) is on or within some 4-sphere of unit radius, and center at a lattice point the sum of whose coordinates is even.

**Proof.** By the lemma each point of \(E_4\) is on or within two spheres of unit radius, and centers at lattice points the sum of whose coordinates differ by one. Thus if we remove from \(S_4\) all spheres with centers of odd coordinate sum, the remaining spheres of \(S_4\) still cover \(E_4\).

3. **Proof of the theorem.** Let \(n \geq 4\) (if \(n = 1, 2, 3\) the theorem is obviously true), and

1) \((x_1, \cdots, x_n)\) be a point in \(E_n\),
2) \(\delta_i = x_i - \left(\left[x_i\right] + 1/2\right) (i = 1, \cdots, n),\)
3) \(\theta_i = \{x_{4i-3}, x_{4i-2}, x_{4i-1}, x_{4i}\} (i = 1, \cdots, m)\) where \(m = \lfloor n/4 \rfloor,\)
4) \(\theta_{m+1} = \{x_{4m+1}, x_{4m+2}, \cdots, x_n\} (\theta_{m+1} \text{ having at most three elements}).\)

We shall choose integers \(y_1, \cdots, y_n\) satisfying the conditions of the theorem.

Let us consider any \(\theta_i\) \((1 \leq i \leq m)\). Let \((i_1, i_2, i_3, i_4)\) be a rearrangement of \((4i-3, 4i-2, 4i-1, 4i)\) so that

\[|\delta_{i_1}| \leq |\delta_{i_2}| \leq |\delta_{i_3}| \leq |\delta_{i_4}|.\]

For \(j = 2, 3, 4\) let

\[(11) z_{ij} = \begin{cases} \left[x_{i_j}\right] & \text{if } \delta_{i_j} \leq 0, \\ \left[x_{i_j}\right] + 1 & \text{if } \delta_{i_j} > 0. \end{cases}\]

By the lemma if we choose \(z_{i_j}\) as either \([x_{i_j}]\) or \([x_{i_j}] + 1\), then in either case

\[(12) \sum_{j=1}^{4} (x_{i_j} - z_{i_j})^2 \leq 1.\]

An inequality (12) holds for each \(\theta_i\) \((i = 1, \cdots, m)\). Therefore

\[(13) \sum_{i=1}^{m} (x_i - z_i)^2 \leq m.\]
Now let $z_{i+1}, \ldots, z_n$ be chosen as in (11), i.e. by replacing $i$ in (11) by $4m + 1, \ldots, n$ successively. Therefore by (4) and (5)

\begin{equation}
\sum_{i=4m+1}^{n} (x_i - z_i)^2 \leq \frac{n - 4m}{4}.
\end{equation}

Therefore by (13) and (14)

\begin{equation}
\sum_{i=1}^{n} (x_i - z_i)^2 \leq \frac{n}{4}
\end{equation}

for any $(z_1, \ldots, z_n)$ which may be chosen.

Associated with each point $(x_1, \ldots, x_n)$ of $E_n$ there is a set of $2^m$ possible lattice points $(z_1, \ldots, z_n)$ which may be selected as in the previous paragraph; let us denote this set by $Z$. Let $(z_1', \ldots, z_l')$ and $(z_1'', \ldots, z_n'')$ be elements of $Z$; we shall call them equivalent if $\sum_{i=1}^{n} (z_i') = \sum_{i=1}^{n} (z_i'')$. Furthermore, for each element $k$ of the set \{0, 1, 2, \ldots, m\} there exist elements $(z_1^*, \ldots, z_n^*)$ and $(z_1^{**}, \ldots, z_n^{**})$ of $Z$ such that

\begin{equation}
\sum_{i=1}^{n} (z_i^*) - \sum_{i=1}^{n} (z_i^{**}) = k.
\end{equation}

Therefore $Z$ can be expressed as a sum of mutually exclusive subsets $Z_1, \ldots, Z_{m+1}$ so that (i) any two elements of $Z_i$ are equivalent, and (ii) $n_i = n_{i-1} + 1$ ($i = 2, \ldots, m+1$) (where $n_i$ denotes the sum of the coordinates of any element of $Z_i$). As $n_1, \ldots, n_{m+1}$ are $m+1$ consecutive integers, it must contain one, say $n_j$, which is divisible by $m+1$. Let $(y_1, \ldots, y_n)$ be any element of $Z_j$. Then (1) is true. This completes the proof of the theorem.

4. Some further questions. It is a corollary of the above proof that there are at least $2^{[n/4]}$ spheres with radii of length $n^{1/2}/2$, and centers at lattice points, which contain $(x_1, \ldots, x_n)$ on or within them. For, (12) holds whether $z_i$ of $Q_i$ ($i = 1, \ldots, [n/4]$) is $[x_{i+1}]$ or $[x_{i+1}] + 1$ ($z_i$, $z_i'$, $z_i''$ of $Q_i$ are defined by (11)). (This yields a constructive process for obtaining all $2^{[n/4]}$ of the spheres mentioned above.) Therefore every $n$-sphere (independent of position) with radius of length $n^{1/2}/2$ must contain at least $2^{[n/4]}$ lattice points on or within it. The length $n^{1/2}/2$ of the radius is “sharp” with respect to the property of “effective lattice point inclusion” as spheres with radii less than $n^{1/2}/2$ do not necessarily have to contain lattice points on or within them (e.g. those with center at $(1/2, 1/2, \ldots, 1/2)$). The following questions remain open:

I. What is the largest number $N(n)$ so that every $n$-sphere (inde-
I. Is it possible (for certain \( n \)) to replace the number \( \lfloor n/4 \rfloor + 1 \) of the theorem by a number \( M(n) \) which is greater than \( \lfloor n/4 \rfloor + 1 \)?

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ON THE MULTIPLICATIVE GROUP OF A DIVISION RING

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Let \( K \) be a noncommutative division ring with center \( Z \) and multiplicative group \( K^* \). Hua [2; 3] proved that (i) \( K^*/Z^* \) is a group without center, and (ii) \( K^* \) is not solvable. A generalization (Theorem 1) will be given here which contains as a special case (Theorem 2) the fact that \( K^*/Z^* \) has no Abelian normal subgroups. This latter theorem obviously contains both (i) and (ii). As a further corollary it is shown that if \( M \) and \( N \) are normal subgroups of \( K^* \) not contained in \( Z^* \), then \( M \cap N \) is not contained in \( Z^* \). The final theorem is that an element \( x \) outside \( Z \) contains as many conjugates as there are elements in \( K \). This makes more precise a theorem of Herstein [1], who showed that \( x \) has an infinite number of conjugates.

Square brackets will denote multiplicative commutation. If \( S \) is a set, then \( o(S) \) will mean the number of elements in \( S \). A subgroup \( H \) of \( K^* \) is subinvariant in \( K^* \) if there is a chain \( \{ N_i \} \) of subgroups such that \( H \triangleleft N_{r-1} \triangleleft \cdots \triangleleft N_1 \triangleleft K^* \), where \( A \triangleleft B \) means that \( A \) is a normal subgroup of \( B \).

Lemma. Let \( K \) be a division ring, \( H \) a nilpotent subinvariant subgroup of \( K^* \), \( y \in H \), \( x \in K^* \), and \( [y, x] = \lambda \in Z^* \), \( \lambda \neq 1 \). Then the field \( Z(x) \) is finite.

Proof. The proof of this lemma is essentially part of Hua’s proof of (ii), but will be included for the sake of completeness.

Let \( f \) be any rational function over \( Z \) such that \( f(x) \neq 0 \). Then

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