ON THE SOLUTION OF $f(f(z)) = e^z - 1$ AND ITS DOMAIN OF REGULARITY

ROBERT OSSERMAN

The problem of examining the complex solutions $f(z)$ of the equation

\[(1) \quad f(f(z)) = e^z - 1\]

was suggested to the author by S. Chowla who, together with Kempner, Rivlin, and Thron, proved that $f(z)$ cannot be an entire function. This result is contained as a special case in two recent papers [1; 3] using the theory of entire functions. The purpose of the present note is to show how the use of elementary geometric methods leads quite easily to a slightly stronger theorem which gives some information on a maximum domain of regularity for $f(z)$. This method may also be applied to the equation

\[(2) \quad f(f(z)) = e^z\]

which was treated in detail by Kneser [2] who proved the existence of a solution analytic on the whole real axis.

**Theorem.** Let $z = x + iy$ and let $\Omega$ denote an infinite strip $|y| < b$ for some constant $b > \pi$. Let $f(z)$ be a function defined in some domain $D$ such that $\Omega \subset D$ and $f(\Omega) \subset D$. If $f(z)$ satisfies (1) throughout $D$ then it cannot be analytic in $\Omega$.

**Proof.** Denote by $R$ the image region under $w = f(z)$ of the strip $S: |y| < \pi$. Then if $\xi = f(\omega) = f(f(z)) = e^z - 1$, the image under $f(\omega)$ of $R$ must be the region $T$ consisting of the $\xi$-plane slit along the negative real axis from $-1$ to $-\infty$. Since the composed map $\xi = f(f(z))$ is a one-one map of $S$ onto $T$, the maps $f$ of $S$ onto $R$ and of $R$ onto $T$ must also be one-one. Furthermore, since the given correspondences between $S$ and $R$ and between $S$ and $T$ are conformal, so is the correspondence between $R$ and $T$.

We note next that $f(0) = 0$. Namely, if $f(0) = a$, then $f(a) = f(f(0)) = 0$, so that $a = f(0) = f(f(a)) = e^a - 1$. But evaluating the derivative of equation (1) at the points 0 and $a$ respectively, we find $f'(a)f'(0) = 1$ and $f'(0)f'(a) = e^a = a + 1$, whence $a = 0$.

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We may thus write the power expansion for \( f(z) \) at the origin in the form \( f(z) = a_1 z + a_2 z^2 + \cdots \), and inserting this in equation (1) shows that \( a_1 = 1 \) and all the \( a_n \) are real. In other words, \( f'(0) = 1 \) and \( f(z) \) is real on the real axis. But the image of the negative x-axis cannot be the whole negative axis in the \( w \)-plane, since the composed map \( f(f(z)) \) must take the negative x-axis onto the interval \((-1, 0)\). Hence \( w = f(z) \) must map the negative x-axis onto an interval \((-c, 0)\) such that \( f: (-c, 0) \to (-1, 0) \). This implies that \( c > 1 \).

We can now show explicitly that the function \( f(z) \) must have a singularity at the point \( z_0 = \log (c - 1) + i\pi \) of the region \( \Omega \). To see this, denote by \( R' \) a copy of the region \( R \) placed in the \( \zeta \)-plane. Then we can find a sequence of points \( \zeta_n \) in the upper half \( \zeta \)-plane such that \( \zeta_n \in R' \cap T \) for all \( n \), and \( \zeta_n \to -c \). Then the corresponding points \( z_n \) of \( S \) such that \( \zeta_n = f(f(z_n)) \) will approach \( z_0 \). On the other hand, their images \( w_n = f(z_n) \) must approach \(-\infty\). To see this we need only note that \( w_n = f^{-1}(\zeta_n) \), so that if we consider the points \( z_n' = w_n \) of the \( z \)-plane and the points \( w_n' = \zeta_n \) of the \( w \)-plane, then \( z_n' = f^{-1}(w_n') \), and \( w_n' \to -c \) implies \( z_n' \to -\infty \).

**Bibliography**

