PSEUDO LOCALLY COMPACT SPACES[1]

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1. Introduction. When metrics are defined in the usual ways for either the set of all functions analytic on a closed region $\bar{R}$ or the set of all functions analytic in an open region R, the spaces so determined have properties resembling local compactness.

For the set of functions analytic on a closed region $\bar{R}$, that is analytic at each point of $\bar{R}$, a metric $d(f, g)$ is usually defined as $\max_{z\in\bar{R}} |(f(z)-g(z)|$. Then any infinite subset of a closed bounded neighborhood $\bar{N}_g$ contains a sequence $\{f_n(z)\}$ which converges uniformly on any closed subset of R [3, p. 140; 4, p. 169][2] and $\lim_{n\to\infty} f_n(z) = f(z)$ is analytic at interior points of R. But the convergence need not be uniform on $\bar{R}$ and $f(z)$ is not necessarily analytic on the closed region $\bar{R}$. Hence, f need not belong to $\bar{N}_g$ and the space is not locally compact.

A metric for the set $\mathcal{R}$ of functions analytic in an open region R is sometimes defined in the following manner [3, p. 139]. Let $\{R_i\}$ be a monotone increasing sequence of R-covering sets and define

$$\rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right) \frac{d_n(f, g)}{1 + d_n(f, g)},$$

where

$$d_n(f, g) = \max_{z\in\bar{R}_n} |f(z) - g(z)|.$$

Let $\mathcal{R}'$ denote the topology induced in $\mathcal{R}$ by the metric ρ and designate the topological space thus defined by $(\mathcal{R}, \mathcal{R}')$. Now, a sequence $\{f_n\}$ in $(\mathcal{R}, \mathcal{R}')$ converges if and only if $\{f_n(z)\}$ converges uniformly on every compact set $K\subset R$ [3, p. 139]. The space $(\mathcal{R}, \mathcal{R}')$ is complete [3, p. 139]. It also has a property which is close to local compactness [3, p. 142], since any infinite uniformly bounded set of functions $\{f_\alpha(z)\}$ contains a sequence $\{f_n(z)\}$ which converges uniformly on any closed subset of R and $\lim_{n\to\infty} f_n(z)$ is analytic in R. However, the

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

sets whose compactness is established are not the closures of neighborhoods by the $\mathcal{R}'$-topology.

In this paper $\mathcal{R}'$-pseudo local compactness is defined. The space $(\mathcal{R}, \mathcal{R})$ is shown to be $\mathcal{R}'$-pseudo locally compact, where $\mathcal{R}$ designates the set of functions analytic in an open region R, $\mathcal{R}$ is the topology induced by the metric $d(f, g)=\sup_{z\in R} |f(z)-g(z)|$, and $\mathcal{R}'$ is the topology defined in the preceding paragraph.

2. **$\mathcal{R}'$-pseudo local compactness defined.** Suppose A is a point set for which two topologies $\mathcal{U}$ and $\mathcal{U}'$ are defined. Thus, topological spaces $(A, \mathcal{U})$ and $(A, \mathcal{U}')$ are determined. If for every point $p\in A$ and an arbitrary $\mathcal{U}'$-neighborhood $N_p(A, \mathcal{U}')$ of p there exists an $\mathcal{U}$-neighborhood $N_p(A, \mathcal{U})\subset N_p(A, \mathcal{U}')$, then $\mathcal{U}$ is said to be *stronger than the* $\mathcal{U}'$*-topology* $(\mathcal{U}\supset\mathcal{U}')$ or $\mathcal{U}'$ *weaker than the* $\mathcal{U}$*-topology* $(\mathcal{U}'\subset\mathcal{U})$ [**2**, p. 16]. We note that if the $\mathcal{U}$-topology is stronger a limit point by the $\mathcal{U}$-topology is necessarily a limit point by the $\mathcal{U}'$-topology.

In this paper, a topological space T is understood to be *compact* if and only if every infinite subset of T has at least one limit point in T. The following well known theorem is stated for reference.

THEOREM A. *If a point set A is compact by a topology $\mathcal{U}$, it is compact by any weaker $\mathcal{U}'$-topology* [**2**, p. 16].

The closure in the space $(A, \mathcal{U})$ by the $\mathcal{U}$-topology of a point set $E\subset A$ will be denoted by $\overline{E}(A, \mathcal{U})$, or in cases where the topology and space are evident simply by $\overline{E}$.

A topological space $(R, \mathcal{R})$ will be said to be $\mathcal{R}'$*-pseudo locally compact*, or *pseudo locally compact by the* $\mathcal{R}'$*-topology*, if any neighborhood by the $\mathcal{R}$-topology of an arbitrary point p of R contains some neighborhood $N_p(R, \mathcal{R})$ by the $\mathcal{R}$-topology, such that the point set $\overline{N}_p(R,\mathcal{R})$ is compact by the $\mathcal{R}'$-topology.

We note that if a space $(R, \mathcal{R})$ is locally compact then $(R, \mathcal{R})$ is $\mathcal{R}$-pseudo locally compact and hence $\mathcal{R}'$-pseudo locally compact by any weaker $\mathcal{R}'$-topology. (See Theorem A.) However, a space $(R, \mathcal{R})$ may be $\mathcal{R}'$-pseudo locally compact but not locally compact by a stronger $\mathcal{R}$-topology. (See paragraph 4 and the example in the next paragraph.) If there are two metric topologies $\mathcal{R}'$ and $\mathcal{R}''$ by which $(R, \mathcal{R})$ is pseudo locally compact and if $\mathcal{R}''\supset\mathcal{R}'$, then $\mathcal{R}'$ and $\mathcal{R}''$ are locally equivalent with respect to $(R, \mathcal{R})$. (See Corollary, Theorem 2.)

We consider an example of an $\mathcal{R}'$-pseudo locally compact space.[3] Let R be the complex plane. For the $\mathcal{R}$-topology define a neighbor-

[3] This example was suggested by Professor R. H. Bing of the University of Wisconsin.

hood of a point p as a circular disc with p as center except that all points, other than p itself, having rational coordinates are omitted. Let $\mathfrak{R}'$ be the topology in which a neighborhood of p is a circular disc with p as center. The space $(R, \mathfrak{R})$ is $\mathfrak{R}'$-pseudo locally compact, but is not locally compact.

3. **General theorems on $\mathfrak{R}'$-pseudo locally compact spaces.** In the following theorems it is to be understood that the $\mathfrak{R}$ or $\mathfrak{R}'$-topology for a subset of R is the relative topology.

We obtain a sufficient condition that a subset of an $\mathfrak{R}'$-pseudo locally compact space be $\mathfrak{R}'$-pseudo locally compact.

THEOREM 1. *Let R be a point set with a topology $\mathfrak{R}$ assigned, also a metric topology $\mathfrak{R}' \subset \mathfrak{R}$.*[4] *Suppose that $(R, \mathfrak{R})$ is $\mathfrak{R}'$-pseudo locally compact. Let R^* be any subset of R. Then $(R^*, \mathfrak{R})$ is $\mathfrak{R}'$-pseudo locally compact if $(R^*, \mathfrak{R}')$ is complete by the $\mathfrak{R}'$-topology.*

PROOF. If $N_q(R^*, \mathfrak{R})$ is any neighborhood of an arbitrary point q of R^*, $N_q(R^*, \mathfrak{R}) = N_q(R, \mathfrak{R}) \cap R^*$. By the hypothesis that $(R, \mathfrak{R})$ is $\mathfrak{R}'$-pseudo locally compact, $N_q(R, \mathfrak{R})$ contains a neighborhood $N_q'(R, \mathfrak{R})$ such that $\overline{N}_q'(R, \mathfrak{R})$ is compact by the $\mathfrak{R}'$-topology. We note that $N_q'(R, \mathfrak{R}) \cap R^*$ is a neighborhood $N_q'(R^*, \mathfrak{R})$ of q in R^*.

We show that $\overline{N}_q'(R^*, \mathfrak{R})$ is compact by the $\mathfrak{R}'$-topology. Let $\{p_r\}$ be any infinite subset of $\overline{N}_q'(R^*, \mathfrak{R})$. Then $\{p_r\} \subset \overline{N}_q'(R, \mathfrak{R})$ and hence by the $\mathfrak{R}'$-topology has a limit point $p \in \overline{N}_q'(R, \mathfrak{R})$. There is a sequence $\{p_{r_i}\} \subset \{p_r\}$ which converges to p by the $\mathfrak{R}'$-topology. Since $(R^*, \mathfrak{R}')$ is complete, $p \in R^*$. Thus, we obtain that $p \in \overline{N}_q'(R, \mathfrak{R}) \cap R^*$, which is just $\overline{N}_q'(R^*, \mathfrak{R})$.

This completes the proof that an arbitrary neighborhood $N_q(R^*, \mathfrak{R})$ of q contains a neighborhood $N_q'(R^*, \mathfrak{R})$ such that $\overline{N}_q'(R^*, \mathfrak{R})$ is compact by the $\mathfrak{R}'$-topology, that is, that $(R^*, \mathfrak{R})$ is $\mathfrak{R}'$-pseudo locally compact.

The following theorem is used in the proof of Theorem 2.

THEOREM B. *For topologies $\mathfrak{R}_\mu \subset \mathfrak{R}_\nu$ on a space R, if R is compact by $\mathfrak{R}_\nu$ and a T_2-space by $\mathfrak{R}_\mu$, then $\mathfrak{R}_\mu$ is equivalent to $\mathfrak{R}_\nu$ (if $\mathfrak{R}_\nu$ is metric)*[5] [2, p. 28].

Let $\mathfrak{R}$, $\mathfrak{R}'$, and $\mathfrak{R}''$ be topologies defined for a point set R. Then $\mathfrak{R}'$ and $\mathfrak{R}''$ are said to be *locally equivalent with respect to* $(R, \mathfrak{R})$ provided the following condition is satisfied: Every neighborhood by

[4] Because of differences in usage of the terms *weaker* and *stronger topologies* the reader is reminded of the definitions in ¶2.

[5] With the additional requirement that the topology is metric our definition of compactness is equivalent to that used by Nakano.

the $\mathfrak{R}$-topology of an arbitrary point p contains a neighborhood $N_p(R, \mathfrak{R})$ such that the relative $\mathfrak{R}'$, and $\mathfrak{R}''$-topologies for the set $S=\overline{N}_p(R, \mathfrak{R})$ (where the closure is by the $\mathfrak{R}$-topology) are equivalent.

THEOREM 2. *Suppose $(R, \mathfrak{R})$ is $\mathfrak{R}'$-pseudo locally compact. Any metric topology $\mathfrak{R}''\supset\mathfrak{R}'$ such that $(R, \mathfrak{R})$ is $\mathfrak{R}''$-pseudo locally compact is locally equivalent to $\mathfrak{R}'$ if $\mathfrak{R}'$ is T_2; any T_2-topology $\mathfrak{R}''\subset\mathfrak{R}'$ is locally equivalent to $\mathfrak{R}'$ if $\mathfrak{R}'$ is metric.*

PROOF. We first consider the case $\mathfrak{R}''\supset\mathfrak{R}'$. Since $(R, \mathfrak{R})$ is $\mathfrak{R}''$-pseudo locally compact, any neighborhood by the $\mathfrak{R}$-topology of a point p contains a neighborhood $N_p(R, \mathfrak{R})$ such that the point set $S=\overline{N}_p(R, \mathfrak{R})$ is compact by the $\mathfrak{R}''$-topology. That is, the space $(S, \mathfrak{R}'')$ is compact. The hypothesis that $(R, \mathfrak{R}')$ is T_2 implies that the subspace $(S, \mathfrak{R}')$ is T_2 by the relative $\mathfrak{R}'$-topology. Now Theorem B implies that the relative $\mathfrak{R}'$ and $\mathfrak{R}''$ topologies for S are equivalent. We have shown that any neighborhood of an arbitrary point p contains a neighborhood $N_p(R, \mathfrak{R})$ such that the relative $\mathfrak{R}'$ and $\mathfrak{R}''$ topologies for the set $S=\overline{N}_p(R, \mathfrak{R})$ are equivalent. That is, $\mathfrak{R}'$ and $\mathfrak{R}''$ are locally equivalent with respect to $(R, \mathfrak{R})$.

Next we consider the case $\mathfrak{R}''\subset\mathfrak{R}'$, where $\mathfrak{R}''$ is T_2. Any neighborhood of an arbitrary point p contains a neighborhood $N_p(R, \mathfrak{R})$ such that $S=\overline{N}_p(R, \mathfrak{R})$ is compact by the metric $\mathfrak{R}'$-topology. The hypothesis that $(R, \mathfrak{R}'')$ is T_2 implies that the subspace $(S, \mathfrak{R}'')$ is T_2. Now Theorem B implies that the relative $\mathfrak{R}'$ and $\mathfrak{R}''$ topologies for S are equivalent. This completes the proof that any neighborhood contains a neighborhood $N_p(R, \mathfrak{R})$ such that the relative $\mathfrak{R}'$ and $\mathfrak{R}''$ topologies for the set $S=\overline{N}_p(R, \mathfrak{R})$ are equivalent, that is, that $\mathfrak{R}'$ and $\mathfrak{R}''$ are locally equivalent with respect to $(R, \mathfrak{R})$.

COROLLARY. *If there are two metric topologies $\mathfrak{R}'$ and $\mathfrak{R}''$, one stronger than the other, by which $(R, \mathfrak{R})$ is pseudo-locally compact, then $\mathfrak{R}'$ and $\mathfrak{R}''$ are locally equivalent with respect to $(R, \mathfrak{R})$.*

4. $\mathfrak{R}'$-pseudo locally compact spaces in analytic function theory. Let R be an open region of the complex plane. Then a sequence of sets $\{R_j\}$ which satisfy the following conditions will be called a *monotone increasing sequence of R-covering sets*: (1) $R_n\subset R$; (2) $\overline{R}_n$ is interior to R_{n+1}; (3) $\bigcup_{j=1}^{\infty} R_j=R$.

Suppose that R is an open region of the complex plane and that $\{R_j\}$ is a particular monotone increasing sequence of R-covering sets. Let $\mathcal{S}$ denote a set of functions which are meromorphic in R.

For each pair of functions f and g of $\mathcal{S}$ let I_{fg} denote the subset of R on each point of which $f(z)$ or $g(z)$ has a pole. Then define a distance function $d(f, g)$ as $\sup_{z\in(R-I_{fg})} |f(z)-g(z)|$ for those $f, g \in \mathcal{S}$ for which this is finite. Now let $\mathcal{R}$ denote the topology so induced in $\mathcal{S}$. We note that if $d(f, g)$ is defined, the poles of $f(z)$ and $g(z)$ in R necessarily coincide and have the same principal parts. Hence, any neighborhood, also any closed neighborhood, of f by the $\mathcal{R}$-topology contains only functions meromorphic in R which have poles in R identical with those of $f(z)$. Thus, if $g\in N_f(\mathcal{S}, \mathcal{R}), f(z)-g(z)$ is analytic in R.

A different topology $\mathcal{R}'$ is induced by the metric function

$$\rho(f, g) = \sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)\frac{d_n(f, g)}{1+d_n(f, g)},)$$

where $d_n(f, g)=\sup_{z\in(R_n-I_{fg})} |f(z)-g(z)|$ [3, p. 139]. Now

$$\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)\frac{d_n(f, g)}{1+d_n(f, g)} \leq \sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)\frac{d(f, g),}{1+d_n(f, g)} \leqq d(f, g);$$

that is, $\rho(f, g)\leqq d(f, g)$. (If $d_n(f, g)$ is undefined, $d_n(f, g)/(1+d_n(f, g))$ is to be taken as "1" in

$$\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)\frac{d_n(f, g)}{1+d_n(f, g)}\cdot)$$

LEMMA. *The $\mathcal{R}$-topology is stronger than the $\mathcal{R}'$-topology.*

PROOF. Corresponding to any point f of $\mathcal{S}$ and to any $\epsilon>0$, there is defined an ϵ-neighborhood $N_f^{(\epsilon)}(\mathcal{S}, \mathcal{R}')$ by the $\mathcal{R}'$-topology and also an ϵ-neighborhood $N_f^{(\epsilon)}(\mathcal{S}, \mathcal{R})$ by the $\mathcal{R}$-topology. We show that $N_f^{(\epsilon)}(\mathcal{S},\mathcal{R}) \subset N_f^{(\epsilon)}(\mathcal{S}, \mathcal{R}')$. If $g\in N_f^{(\epsilon)}(\mathcal{S}, \mathcal{R})$, $d(f, g)<\epsilon$. Since $\rho(f, g)\leqq d(f, g)$, $\rho(f, g)<\epsilon$ and so $g\in N_f^{(\epsilon)}(\mathcal{S}, \mathcal{R}')$.

The following theorem is similar to one stated by Thron [3, p. 139] for analytic functions. We shall say that a sequence $\{f_n(z)\}$ of functions meromorphic on $\overline{R}_k$ converges uniformly to $f(z)$ on $\overline{R}_k$, if, corresponding to any $\epsilon>0$, there exists N such that $n>N$ implies $|f(z)-f_n(z)|<\epsilon$ on $(\overline{R}_k-I_{f,f_n})$.

THEOREM C. *Let $\mathcal{S}$ be any infinite set of functions which are meromorphic in an open region R. A sequence $\{f_n\}$ in $(\mathcal{S}, \mathcal{R}')$ converges to f in $(\mathcal{S}, \mathcal{R}')$ if and only if $\{f_n(z)\}$ converges uniformly to $f(z)\in\mathcal{S}$ on every $\overline{R}_k$.*

PROOF. Suppose $\{f_n\}$ converges to f in $(\mathcal{S}, \mathcal{R}')$, that is, that $\lim_{n\to\infty} \rho(f_n, f)$, or

$$\lim_{n\to\infty} \sum_{j=1}^{\infty}\left(\frac{1}{2^j}\right)\frac{d_j(f_n, f)}{1+d_j(f_n, f)} = 0,$$

where $f \in \mathcal{S}$. It is sufficient to show that this implies $\lim_{n\to\infty} d_k(f_n, f) = 0$ for every k.

We first show that, for arbitrary k,

$$\lim_{n\to\infty} \frac{d_k(f_n, f)}{1 + d_k(f_n, f)} = 0.$$

If

$$\lim_{n\to\infty} \frac{d_k(f_n, f)}{1 + d_k(f_n, f)} \neq 0,$$

then for some $\eta > 0$ and every N there is $r > N$ such that

$$\frac{d_k(f_r, f)}{1 + d_k(f_r, f)} > \eta.$$

Then

$$\rho(f_r, f) = \sum_{j=1}^{\infty} \left(\frac{1}{2^j}\right) \frac{d_j(f_r, f)}{1 + d_j(f_r, f)} > \frac{\eta}{2^k} \cdot$$

Thus, for any N, there exists $r > N$ such that $\rho(f_r, f) > \eta/2^k$. This contradicts the hypothesis that $\lim_{n\to\infty} \rho(f_n, f) = 0$. We conclude that, for arbitrary k,

$$\lim_{n\to\infty} \frac{d_k(f_n, f)}{1 + d_k(f_n, f)} = 0,$$

and hence that $\lim_{n\to\infty} d_k(f_n, f) = 0$, that is, that $\{f_n(z)\}$ converges to $f(z)$ uniformly on each $\overline{R}_k$.

To prove the converse we note that, if $\{f_n(z)\}$ converges uniformly to $f(z)$ on every $\overline{R}_k$, then $\lim_{n\to\infty} (d_k(f_n, f)/(1 + d_k(f_n, f))) = 0$. There exists $N(k)$ so that $n > N(k)$ implies that

$$\frac{d_k(f_n, f)}{1 + d_k(f_n, f)} < \frac{\epsilon}{2^k} \cdot$$

Then

$$\begin{aligned} \rho(f_n, f) &< \sum_{j=1}^{k} \left(\frac{1}{2^j}\right) \frac{\epsilon}{2^k} + \sum_{j=k+1}^{\infty} \frac{1}{2^j} \\ &= \frac{2^k - 1}{2^k} \left(\frac{\epsilon}{2^k}\right) + \frac{1}{2^k} \qquad \text{for all } n > N(k). \end{aligned}$$

Thus, corresponding to any $\eta > 0$, k can be chosen so that when

$n>N(k)$, $\rho(f_n, f)<\eta$. Hence, $\lim_{n\to\infty} \rho(f_n, f)=0$, or $\lim_{n\to\infty} f_n=f$ in $(\mathcal{S}, \mathfrak{R}')$.

THEOREM 3. *Let* $\mathfrak{R}$ *be the set of all functions meromorphic in an open region* R. *Then* $(\mathfrak{R}, \mathfrak{R})$ *is* $\mathfrak{R}'$*-pseudo locally compact.*

PROOF. Let f be any point of R; that is, let $f(z)$ be any function which is meromorphic in R. An ϵ-neighborhood of f by the $\mathfrak{R}$-topology is defined by

$$N_f(\mathfrak{R}, \mathfrak{R}) = \{g(z) \mid (1)\ g(z) \in \mathfrak{R} \text{ and } (2)\ d(f, g) < \epsilon\}.$$

Then

$$\overline{N}_f(\mathfrak{R}, \mathfrak{R}) = \{g(z) \mid (1)\ g(z) \in \mathfrak{R} \text{ and } (2)\ d(f, g) \leqq \epsilon\}.$$

It is sufficient to show that any infinite subset $\{g_\alpha\}$ in $\overline{N}_f(\mathfrak{R}, \mathfrak{R})$ has a limit point by the $\mathfrak{R}'$-topology. We note that each function $g_\alpha(z)$ necessarily has poles coinciding with those of $f(z)$ in R with the same principal parts as those of $f(z)$. Hence, $f(z)-g_\alpha(z)$ is analytic everywhere in R. We define $h_\alpha(z)=f(z)-g_\alpha(z)$ when $z\in R$. Since $|h_\alpha(z)| \leqq \epsilon$ when $z\in R$, $\{h_\alpha(z)\}$ is uniformly bounded on R. We have shown that $\{h_\alpha(z)\}$ is a set of functions analytic and uniformly bounded on R. Therefore, Ascoli's Theorem [**4**, p. 169; **3**, p. 140] implies the existence of a sequence $\{h_{\alpha_n}(z)\}$ which is a subset of $\{h_\alpha(z)\}$ and which converges at each point of R, uniformly on each $\overline{R}_n$. By a well known theorem [**4**, p. 98] $\lim_{n\to\infty} h_{\alpha_n}(z)=h(z)$ is analytic in R. Moreover, since $|h_{\alpha_n}(z)| = |f(z)-g_{\alpha_n}(z)| \leqq \epsilon$, $|h(z)| \leqq \epsilon$.

Define $g(z)=f(z)-h(z)$. We note that, in R, $h(z)=\lim_{n\to\infty} h_{\alpha_n}(z) =\lim_{n\to\infty} (f(z)-g_{\alpha_n}(z)) =f(z)-\lim_{n\to\infty} g_{\alpha_n}(z)$, where the convergence holds everywhere in R (except at the points of I_{fg}) and uniformly on every $\overline{R}_n$. That is, $h(z)=f(z)-\lim_{n\to\infty} g_{\alpha_n}(z)$. Then, since $h(z) =f(z)-g(z)$, we conclude that $g(z)=\lim_{n\to\infty} g_{\alpha_n}(z)$; hence, Theorem C implies that $\{g_{\alpha_n}\}$ converges to g in $\mathfrak{R}$ by the $\mathfrak{R}'$-topology. As was noted in the preceding paragraph, $|h(z)| \leqq \epsilon$; that is, $|f(z)-g(z)| \leqq \epsilon$. Then, since $g(z)$ is meromorphic in R and, therefore, belongs to $\mathfrak{R}$, $g\in \overline{N}_f(\mathfrak{R}, \mathfrak{R})$.

Thus, if f is any point of $\mathfrak{R}$ and $N_f(\mathfrak{R}, \mathfrak{R})$ is any neighborhood of f by the $\mathfrak{R}$-topology, $N_f(\mathfrak{R}, \mathfrak{R})$ contains a neighborhood—namely, $N_f(\mathfrak{R}, \mathfrak{R})$ itself—such that any infinite subset $\{g_\alpha\}$ in $\overline{N}_f(R, \mathfrak{R})$ contains a sequence $\{g_{\alpha_n}\}$ which converges by the $\mathfrak{R}'$-topology to a point $g\in \overline{N}_f(\mathfrak{R}, \mathfrak{R})$. This completes the proof that $(R, \mathfrak{R})$ is $\mathfrak{R}'$-pseudo locally compact.

COROLLARY. *Any metric topology* $\mathfrak{R}''$, *either stronger or weaker than* $\mathfrak{R}'$, *by which* $(\mathfrak{R}, \mathfrak{R})$ *is pseudo locally compact is locally equivalent to* $\mathfrak{R}'$.

PROOF. This follows directly from the corollary of Theorem 2.

In the following theorems, unless indicated otherwise, it is to be understood that R is an open region, $\mathfrak{R}$ is the set of all functions meromorphic in R, and that topologies $\mathfrak{R}$ and $\mathfrak{R}'$ are defined as before.

THEOREM 4. *Let $\mathfrak{R}^*$ be a set of functions meromorphic in R. Then $(\mathfrak{R}^*, \mathfrak{R})$ is $\mathfrak{R}'$-pseudo locally compact if $(\mathfrak{R}^*, \mathfrak{R}')$ is complete.*

PROOF. We note that $\mathfrak{R}^* \subset \mathfrak{R}$. By Theorem 3 $(\mathfrak{R}, \mathfrak{R})$ is $\mathfrak{R}'$-pseudo locally compact and by hypothesis $(\mathfrak{R}^*, \mathfrak{R}')$ is complete. Now Theorem 1 implies that $(\mathfrak{R}^*, \mathfrak{R})$ is $\mathfrak{R}'$-pseudo locally compact.

THEOREM 5. *The subspace $(\mathfrak{R}^{(A)}, \mathfrak{R})$ of $(\mathfrak{R}, \mathfrak{R})$ whose points are those functions analytic everywhere in R is $\mathfrak{R}'$-pseudo locally compact.*

PROOF. According to a theorem stated by Thron [3, p. 139], $(\mathfrak{R}^{(A)}, \mathfrak{R}')$ is complete. The required result follows from Theorem 4.

We next investigate the connection between a normal family of meromorphic functions and a set of meromorphic functions $\mathfrak{R}$ such that $(\mathfrak{R}, \mathfrak{R})$ is $\mathfrak{R}'$-pseudo locally compact. Of course, a normal family $\mathfrak{R}$ of functions meromorphic in a region R can be extended to a compact normal family by adding to $\mathfrak{R}$ all of its limit functions [1, p. 183]. In the next theorem we suppose that $\mathfrak{R}$ is an infinite set of functions meromorphic in a region R and show that if $(\mathfrak{R}, \mathfrak{R})$ is $\mathfrak{R}'$-pseudo locally compact then certain subsets of $\mathfrak{R}$ form normal families of functions.

THEOREM 6. *If $(\mathfrak{R}, \mathfrak{R})$ is $\mathfrak{R}'$-pseudo locally compact, then every neighborhood by the $\mathfrak{R}$-topology of an arbitrary point q of $\mathfrak{R}$ contains a neighborhood $N_q(\mathfrak{R}, \mathfrak{R})$ such that the set of functions $\overline{N}_q(\mathfrak{R}, \mathfrak{R})$ is a normal family of functions.*

PROOF. Choose $N_q(\mathfrak{R}, \mathfrak{R})$ so that $\overline{N}_q(\mathfrak{R}, \mathfrak{R})$ is compact by the $\mathfrak{R}'$-topology. Then every sequence $\{f_n\} \subset \overline{N}_q(\mathfrak{R}, \mathfrak{R})$ has (by the $\mathfrak{R}'$-topology) a limit point $f \in \overline{N}_q(\mathfrak{R}, \mathfrak{R})$. There exists a subsequence $\{f_{n_i}\}$ which converges to f by the $\mathfrak{R}'$-topology; that is, $\{f_{n_i}(z)\}$ converges to $f(z)$ everywhere in R and uniformly on every closed subset of R.

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