MULTIPLIERS ON COMPLEX HOMOGENEOUS SPACES

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1. Let $G$ be a real Lie group, represented as a transitive group of analytic automorphisms of a simply-connected complex analytic manifold $D$; if $g \in G$ and $z \in D$, the action of the transformation representing $g$ on the point $z$ will be denoted by $gz$. A multiplier for the group $G$, with respect to its representation as a transformation group on $D$, is a $C^\infty$ complex-valued function $\mu(g; z)$ on $G \times D$ which is holomorphic in $z$ and which satisfies $\mu(g_1 g_2; z) = \mu(g_1; g_2 z) \mu(g_2; z)$ for every $g_1, g_2 \in G$; to exclude the obvious trivial case, we further assume that $\mu(g; z) \neq 0$. Such functions are sometimes considered in examining the group $G$ and its representations, but also arise as the continuous analogs of some structures of interest in the study of automorphic functions; our purpose here is to determine the possible multipliers which may arise in connection with the second of the above points of view. We shall always assume here that $G$ is connected. Notice that the set of all multipliers for $G$ forms an abelian group $\mathfrak{M}(G; D)$ under multiplication.

The universal covering group $G^*$ of $G$ also acts as a transformation group on $D$, the action of the transformation representing $g^* \in G^*$ on the point $z \in D$ being defined by $g^* z = gz$ whenever $g^*$ covers $g$; we shall consider firstly the group $\mathfrak{M}(G^*; D)$ of multipliers for $G^*$. Let $H^*$ be the isotropy subgroup of $G^*$ at some point $z_0$, which point is to be held fixed subsequently, and let $K^*$ be the subgroup of $G^*$ consisting of all elements represented by the trivial transformation which leaves $D$ pointwise fixed. For our purposes, in particular for Siegel’s modular groups, there is no loss of generality in assuming:

(i) that there are local $C^\infty$ mappings $z \rightarrow g^*_z$ of $D$ into $G^*$ such that $g^*_z z_0 = z$; (ii) that $K^*$ is the center of $G^*$; (iii) that $K^* \cap [G^*, G^*] = e^*$, that is, the intersection of the center and the commutator subgroup of $G^*$ is the trivial subgroup consisting of the identity $e^*$ alone; and (iv) that elements of finite order are everywhere dense in the group $H^*/K^*$.

Whenever $f(z)$ is holomorphic and nowhere vanishing on $D,$
\[ \mu_0(g^*; z) = f(g^*z)f(z)^{-1} \] is a multiplier; these are called the trivial multipliers, and form a subgroup \( \mathfrak{M}(G^*; D) \) of \( \mathfrak{M}(G^*; D) \) which is canonically isomorphic to the group of holomorphic, nowhere-vanishing functions on \( D \). Let \( \mathcal{E}(G^*; D) \) be the additive group of \( G^* \)-invariant differential forms of type \((0, 1)\) on the manifold \( D \) which are of the form \( \bar{\partial}g(z, \bar{z}) \) for some \( C^\infty \) complex valued function \( g(z, \bar{z}) \) on \( D \); in the cases which arise from automorphic functions, when \( D \) is a Stein manifold, these differential forms are just the \( \bar{\partial} \)-closed \( G^* \)-invariant forms of type \((0, 1)\). Finally let \( \text{Hom} \ (K^*; C) \) be the group of \( C^\infty \) homomorphisms of \( K^* \) into the additive group of complex numbers.

**Theorem.** The group \( \mathfrak{M}(G^*; D) \) is canonically isomorphic to a direct sum as follows:

\[ \mathfrak{M}(G^*; D) \cong \mathcal{E}(G^*; D) \oplus \mathfrak{Z}(G^*; D) \oplus \text{Hom} \ (K^*; C). \]

**Proof.** Since \( G^* \times D \) is simply-connected, \( \sigma(g^*; z) = \log \mu(g^*; z) \) is a well-defined single-valued function, that branch of the logarithm being selected for which \( \sigma(e^*; z) = 0 \) for the identity \( e^* \in G^* \); moreover \( \sigma(g_1^*g_2^*; z) = \sigma(g_1^*; g_2^*z) + \sigma(g_2^*; z) \) for every \( g_1^*, g_2^* \in G^* \). In particular, whenever \( k^* \in K^* \) and \( g^* \in G^* \), it follows from assumption (ii) that \( g^* = g_1^*g_2^* = k^* \), and hence that \( \sigma(k^*; z) = \sigma(g^* - k^*g^*; z) = \sigma(g^* - 1; k^*g^*z) + \sigma(k^*; g^*z) + \sigma(g^*; z) = \sigma(k^*; g^*z) \); therefore \( \sigma(k^*; z) = \sigma(k^*) \) is a constant. The mapping \( k^* \mapsto \sigma(k^*) \) is an element of \( \text{Hom} \ (K^*; C) \), and the mapping \( \mu(g^*; z) \mapsto \sigma \) is a homomorphism of \( \mathfrak{M}(G^*; D) \) into \( \text{Hom} \ (K^*; C) \). Now restricting ourselves to the kernel of the above homomorphism in \( \mathfrak{M}(G^*; D) \), we have \( \sigma(k^*; z) = 0 \) for every \( k^* \in K^* \). Whenever \( h^* \in H^* \) corresponds to an element of finite order in \( H^*/K^* \), say \( h^* \in K^* \), then \( 0 = \sigma(h^*; z_0) = n \sigma(h^*; z_0) \); but since such elements are everywhere dense in \( H^* \) by assumption (iv), it follows that \( \sigma(h^*; z_0) = 0 \) for every \( h^* \in H^* \). Thus for every \( g^* \in G^* \), \( g_0^*z_0 = g^*z = g_0^*g^*z_0 \), so that \( g_0^*g_0^*g^* \in H^* \); consequently \( 0 = \sigma(g_0^*g_0^*g^*; z_0) = -\sigma(g_0^*; z_0) + \sigma(g^*g^*; z_0) \). Now \( f(z) = \sigma(g^*; z_0) \) is independent of the choice of local sections \( g_0^* \), is clearly a \( C^\infty \) function on \( D \) by assumption (i), and for any \( g^* \in G^* \), \( f(g^*z) = \sigma(g^*g^*; z_0) = \sigma(g^*g^*; z_0) = \sigma(g^*; z) + f(z) \). Obviously any other function satisfying this functional equation differs from \( f(z) \) at most by an additive constant. To each \( \sigma(g^*; z) \) associate the differential form \( \bar{\partial}f(z) \); this defines a homomorphism of the set of multipliers with \( \sigma = 0 \) into the group \( \mathcal{E}(G^*; D) \), and the kernel is clearly precisely the group \( \mathfrak{M}(G^*; D) \).

To complete the proof, we need merely show that the above homomorphisms are onto. For any differential form \( \bar{\partial}f(z) \in \mathcal{E}(G^*; D) \),
\( \mu(g; z) = \exp(f(gz) - f(z)) \) is a multiplier having \( \theta = 0 \) and mapping onto the form \( \tilde{\theta}f(z) \) by the previous mapping. Further, any element \( \tilde{\theta} \in \text{Hom}(K^*; C) \) can be extended to an element \( \tilde{\theta}_1 \in \text{Hom}(G^*; C) \); for if we map \( G^* \) homomorphically onto the abelianized group \( G^*/[G^*, G^*] \), this will be an isomorphism on \( K^* \) by assumption (iii), and since a homomorphism on a subgroup of an abelian group can be extended to the full group, \( \tilde{\theta} \) clearly admits the desired extension, the exponential of which is a multiplier mapping onto the element \( \tilde{\theta} \). This therefore concludes the proof.

Any homomorphic image \( G_1 \) of \( G^* \) for which the kernel \( K_1^* \) of the homomorphism \( G^* \rightarrow G_1 \) is contained in \( K^* \) likewise acts as a transformation group on \( D \); the multipliers \( \mathfrak{m}(G_1; D) \) are determined by the subgroup of \( \mathfrak{m}(G^*; D) \) consisting of those multipliers \( \mu(g^*; z) \) for which \( m(W; z) = 1 \) whenever \( k^* \in K^* \).

2. As an example, consider the symplectic group acting on the generalized unit disc of degree \( p \), as introduced by Siegel. The assumptions as listed previously are fulfilled in this case. Moreover the group \( \mathfrak{g}(G^*; D) \) is a one-dimensional vector space over the complex numbers. To see this, recall that there is but one independent, closed \( G \)-invariant differential form of type \((1, 1)\) on \( D \) in this case, namely the form \( \Omega \) determined by the metric. If \( \theta_1, \ldots, \theta_a \) form a basis for \( \mathfrak{g}(G^*; D) \), the elements \( \partial \theta_1, \ldots, \partial \theta_a \) must be dependent; hence by a suitable choice of the base, we may assume that \( \partial \theta_a = \cdots = \partial \theta_a = 0 \). But then \( \theta_j / \partial \theta_j \) will be a closed and invariant form for \( j \geq 2 \), and must be zero since \( \Omega \) does not admit such a decomposition; hence \( \theta_j = 0 \) for \( j \geq 2 \), and \( a = 1 \). Therefore in this case, the nonobvious multipliers in \( \mathfrak{m}(G^*; D) \) are powers of the Jacobian determinants of the transformations representing elements of \( G^* \). The same is of course true whenever there is but one closed invariant form of type \((1, 1)\) on \( D \).

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5 This would imply \( \Omega \wedge \Omega = 0 \), which is impossible for a Kaehler metric.