A THEOREM ON POWER SERIES WHOSE COEFFICIENTS HAVE GIVEN SIGNS

W. H. J. FUCHS

1. The following theorem, first proved by A. Hurwitz and G. Pólya, is well known ([3] or [1, p. 99]).

If \( \sum_{k=0}^{\infty} a_k z^k \) is a power series of finite radius of convergence, then it is possible to find a sequence \( \{ \epsilon_k \} (\epsilon_k = \pm 1) \) such that the series \( \sum_{k=0}^{\infty} \epsilon_k a_k z^k \) has the circle of convergence as natural boundary.

In this note I prove the following companion-piece to Pólya's theorem.

**Theorem.** If \( \{ \epsilon_k \}_{k=0}^{\infty} \) is a sequence with \( \epsilon_k = \pm 1 \), then there is always a power series \( \sum a_k z^k \), \( a_k > 0 \), of finite radius of convergence such that the series \( \sum \epsilon_k a_k z^k \) can be analytically continued across a semi-circle on its circle of convergence.

This theorem answers in the negative the question: Is there a "universal scrambling sequence" \( \{ \epsilon_k \} \), \( \epsilon_k = \pm 1 \), turning every power series \( \sum a_k z^k \) with positive coefficients into a power series \( \sum \epsilon_k a_k z^k \) having the circle of convergence as natural boundary? This problem was raised by Mrs. Turán, and I am indebted to Dr. P. Erdős for communicating it to me.

An example (§4) shows that the semi-circle in the statement of the theorem cannot be replaced by a larger arc.

A question which remains open is to find a corresponding theorem for the case that \( \{ \epsilon_k \} \) is a given sequence of complex numbers of absolute value one.

2. The following lemmas are required.

**Lemma 1.** Let \( \Lambda = \{ \lambda_n \} \) be a sequence of positive numbers no two of which are at a distance less than \( c > 0 \) from each other. Let

\[
g(z) = \prod_{\lambda \in \Lambda} \frac{\lambda - z}{\lambda + z} e^{z^2 / \lambda}.
\]

Then there are constants \( A \) and \( B \) such that in \( x \geq 0, |z - \lambda| \geq c / 4 (\lambda \in \Lambda) \)

\[
0 < (B e^{\phi(r)})^2 \leq |g(z)| \leq (A e^{\phi(r)})^2
\]

where \( r = |z| = |x + iy| \) and

\[
\text{Received by the editors April 9, 1956 and, in revised form, August 23, 1956.}
\]

\[
1 \text{ This research was supported by the United States Air Force under Contract No. AF 18(600)-685 monitored by the Office of Scientific Research.}
\]
\[ \phi(r) = \sum_{\lambda < r; \lambda \in \Lambda} 2/\lambda. \]

For a proof of this lemma see [2, Lemmas 3 and 4].

**Lemma 2.** Let \( M = \{\mu\} \) be a sequence of positive numbers whose mutual distances are \( \geq 1 \). Suppose that the function \( h(\xi) = h(\xi + i\eta) \) is regular in the region \( \xi \geq 0, \xi \neq 0 \) except for simple poles at the points \( \xi = \mu \in M \). Suppose further that there are positive constants \( A, \alpha, \beta \) (\( \beta < \pi \)) such that

\[ |h(\xi)| = |h(\xi + i\eta)| < Ae^{-\alpha |\xi|} \]

in \( \xi \geq 0, \xi \neq 0 \), except in circles of radius 1/4 with centers at the points \( \mu \in M \).

Then the function

\[ H(z) = \sum_{\mu \in M} r_{\mu} z^{\mu} \]

is regular in the sector \( 0 < |z|, |\arg z| < \beta \); where \( r_{\mu} \) is the residue of \( h(\xi) \) at \( \mu \).

**Proof.** Let \( C_R \) be the semicircle \( |\xi| = R, \xi \geq 0 \) and let \( L_R \) be a curve with endpoints \( \xi = iR \) and \( \xi = -iR \) which runs along the imaginary axis except for an indentation into the right half plane near \( \xi = 0 \).

By the residue theorem

\[ \frac{1}{2\pi i} \int h(\xi) e^{-k\xi} d\xi = \sum_{\mu \in M} r_{\mu} e^{-k\mu} \]

where the integration is along \( C_R + L_R \). If the number \( k \) is chosen positive and larger than \( \alpha \), then on \( C_R \)

\[ |h(\xi) e^{-k\xi}| < Ae^{-(\alpha - k) |\xi|} \]

and therefore

\[ \int_{C_R} h(\xi) e^{-k\xi} d\xi \to 0 \]

as \( R \to \infty \) through a sequence of values avoiding the intervals \( |\xi - \mu| \) \( < 1/4 \) on the real axis. It follows that for \( z = e^{-k} \) (\( k > \alpha \)) the series

\[ \sum_{\mu \in M} r_{\mu} z^{\mu} \]

converges to a function.
(2) \[ H(z) = \sum r_nz^n = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} h(\zeta)z^\zeta d\zeta, \]

where the path of integration is the imaginary axis with an indentation near \( \zeta = 0 \). For purely imaginary values of \( \zeta \), \( |\zeta| = |\text{Re} \zeta| = e^{-\pi \cdot \arg \zeta} \). Hence, in \( |\arg z| \leq \beta' < \beta \), \( |h(i\eta)z^i\eta| < A e^{-|\beta| |\eta|+|\eta| \cdot \arg z|} < A e^{-(\beta-\beta')|\eta|} \). This shows that the integral on the right-hand side of (2) is uniformly convergent in \( |\arg z| \leq \beta' < \beta \). Therefore it defines the analytic continuation of \( H(z) \) into the whole sector \( |\arg z| < \beta \).

3. Proof of the theorem. Let \( \{\lambda\} = \Lambda \) be the set of those odd multiples of \( 1/2 \) for which \( \epsilon_{\lambda-1/2}\epsilon_{\lambda+1/2} = -1 \). Write

\[ \phi(r) = \sum_{\lambda < r} 2/\lambda, \]

\[ m = \lim \inf \phi(r) - \log r, \]

\[ M = \lim \sup \phi(r) - \log r. \]

We consider separately the five cases:

(i) \( -\infty < M < \infty \). (ii) \( M = -\infty \). (iii) \( -\infty < m < \infty \).

(iv) \( m = \phi \). (v) \( m = -\infty, M = \infty \).

These cases are not mutually exclusive, but they cover all possibilities.

(i) \( -\infty < M < \infty \). Define \( g(z) \) by (1). By Lemma 1

\[ 0 < \limsup_{n \to \infty} |g(n)|^{1/n} = C < \infty \quad (n = 1, 2, \ldots). \]

The function

\[ h(\zeta) = (C\zeta)^{-\imath} g(\zeta) \csc \pi \zeta \]

satisfies the hypotheses of Lemma 2 with \( \beta = \pi/2, M = \{1, 2, 3, \ldots\} \). The residue of \( h(\zeta) \) at \( n \) is

\[ r_n = (-1)^n g(n)/\pi(Cn)^n. \]

By the choice of \( C \), the series

\[ \sum r_n(-z)^n = \sum c_nz^n = f(z) \]

has radius of convergence 1. The sign of \( c_n \) is the same as that of \( g(n) \). But \( g(z) \) changes sign between those integers \( k, k+1 \) for which \( \epsilon_k \) and \( \epsilon_{k+1} \) are of opposite sign and nowhere else. Hence \( \epsilon_n g(n) \) is of constant sign. Also, by Lemma 2, \( f(z) \) is regular in \( |\arg (-z)| < \pi/2, \)
i.e. in \( x < 0 \). Therefore one of the two functions \( \pm f(z) \) has the required properties.

(ii) \( M = -\infty \). We can find a sequence \( \{\nu\} \) of odd multiples of \( 1/2 \) which has no terms in common with \( \Lambda \) and for which \( \lim \sup \{\phi(r) + \sum_{r<r} 4/\nu - \log r\} = 0 \), say. The construction of the previous case can now be used, if \( g(\zeta) \) is replaced by

\[
g(\zeta) \cdot \left\{ \prod_{\nu} \frac{\nu - \zeta}{\nu + \zeta} e^{2\phi/\nu} \right\}^2.
\]

(iii) \( -\infty < m < \infty \). Let \( g(\zeta) \) again be defined by (1). If \( D \) is any positive number, the function

\[
h(\zeta) = (D\zeta)^m / g(\zeta)
\]

satisfies the hypotheses of Lemma 2, with \( \{\mu\} = \{\lambda\} \), \( \beta = \pi/2 \). The residue at \( \lambda = \rho \) is

\[
r_\rho = (D\rho)^m / g'(\rho).
\]

Now

\[
g'(\rho) = - \prod_{\lambda \in \Lambda, \lambda \neq \rho} \frac{\lambda - \rho}{\lambda + \rho} e^{2\phi/\rho} \cdot \frac{e^2}{2\rho}
\]

and so, by Lemma 1 \( g'(\rho) \) lies between

\[
(B_1 e^{\phi(\rho)})^\rho \quad \text{and} \quad (A_1 e^{\phi(\rho)})^\rho,
\]

where \( A_1 \) and \( B_1 \) are independent of \( \rho \). Since \( \lim \inf (\phi(\rho) - \log \rho) \) is finite, the constant \( D \) can be adjusted so that the series

\[
\sum_{\rho \in \Lambda} r_\rho \cdot z^{\rho - 1/2} = \psi(z)
\]

has radius of convergence 1. By Lemma 2 \( \psi(z) \) is regular in \( x > 0 \). The values of \( \epsilon \) at successive terms of the sequence \( \{\rho - 1/2\} \) are of opposite sign, since two such integers are separated by exactly one term of the sequence \( \Lambda \).

The coefficients \( r_\rho \) of two consecutive terms in the power series are also of opposite sign, since the slope of \( g(x) \) has opposite signs at successive zeros \( \rho \) of \( g(x) \).

Therefore \( \pm \psi(z) \) satisfies all requirements, except that it has zero coefficients for all integers which are not of the form \( \rho - 1/2 \), \( \rho \in \Lambda \). By adding to \( \pm \psi(z) \) an entire function whose power series has suitable signs, an example satisfying all requirements is obtained.

\footnote{This is easily verified, if the inequality \( \theta \sin \theta \leq (\pi/2)(1 - \cos \theta) \) is noted.}
(iv) \( m = \infty \). We can choose a sub-sequence of \( \Delta \) such that we have case (iii) for the new sequence and such that any two consecutive terms of the sub-sequence are separated by an even number of terms in the original sequence \( \Delta \). The construction of case 3 can now be applied.

(v) \( m = -\infty, M = \infty \). Put

\[
\gamma(r) = \phi(r) - \log r = \sum_{\lambda < r} 2/\lambda - \log r.
\]

The hypothesis \( m = -\infty, M = \infty \) implies that given \( q > 0 \) there are arbitrarily large numbers \( h, k \) such that

\[
\inf_{h < r < k} \gamma(r) - \gamma(h) = -q
\]

and

\[
0 \leq \gamma(k) - \gamma(h) \leq 1.
\]

A set \( M = \{ \mu \} \) is now constructed as follows.

First a sequence of nonoverlapping intervals \( I_1, I_2, \ldots \) is chosen such that in \( I_q = (h_q, k_q) \) (3) and (4) hold, with \( h = h_q \) and \( k = k_q \), and such that \( h_{q+1} > 10k_q \) \((q = 1, 2, \ldots)\). All terms of \( \Delta \) which lie in \( I_q \) are terms of the new sequence \( M \), in fact \( M \cap I_q = \Delta \cap I_q \) \((q = 1, 2, \ldots)\).

If \( \delta(r) = \sum_{\mu < r, \mu \in M} 2/\mu - \log r \), then it follows that (3) and (4) hold with \( \delta \) in place of \( \gamma \) and \( h = h_q, k = k_q \).

Next we select further positive odd multiples of 1/2 as members of \( M \).

At first enough of these are chosen from the interval \((0, h_1)\) to make

\[
0 < \delta(h_1) \leq 1.
\]

This gives, by (4),

\[
0 \leq \delta(k_1) \leq 2.
\]

Next enough terms are added between \( k_1 \) and \( h_2 \) to make

\[
0 \leq \delta(r) \leq 2 \quad (k_1 \leq r \leq h_2)
\]

and

\[
1 \leq \delta(h_2) \leq 2.
\]

Then, by (3)

\[
1 \leq \delta(k_2) \leq 3
\]

and we make

\[
1 \leq \delta(r) \leq 3 \quad (k_2 \leq r \leq h_3),
\]

\[
2 \leq \delta(h_3) \leq 3.
\]
Continuing in this way we can choose \( M \) so that
\[
q - 2 \leq \delta(r) \leq q \quad (k_{q-1} \leq r \leq h_q; q = 2, 3, \ldots).
\]
Hence \( \delta(r) \to \infty \) as \( r \to \infty \) through any set of values outside the intervals \( I_q \). But by (3)
\[
-1 \leq \lim \inf \delta(r) \leq 0.
\]
Since \( \delta(r) \) has local minima at the \( r \in M \), there is a \( \mu = \mu_q^* \in I_q \) such that
\[
-1 \leq \delta(\mu_q^*) \leq 0; \quad \lim \inf \delta(r) = \lim \inf \delta(\mu_q^*).
\]
Since
\[
\delta(h_q) - \delta(\mu_q^*) = \log \frac{\mu_q^*/h_q}{1 - 2/\mu} \geq q - 2
\]
(5) \( \mu_q^* > 3h_q \) \( (q \geq 4) \).
Similarly
(6) \( \mu_q^* < k_q/3 \) \( (q \geq 4) \).
Now let \( g_1(z) \) be defined by (1) with \( M \) in place of \( \Lambda \). As in case (iii) we derive a function
\[
\psi(z) = \sum_{\mu \in M} r_\mu \mu^{-1/2}
\]
from the auxiliary function \( h(\xi) = (D_\xi)^f/g(\xi) \). This function \( \psi(z) \) has the following properties:
1. Its power series has radius of convergence 1.
2. As \( \mu \to \infty \) through the sequence \( \mu_1^*, \mu_2^*, \ldots \),
\[
\lim \sup |r_\mu|^{1/(\mu-1/2)} = 1.
\]
3. \( \psi(z) \) is regular in \( x > 0 \).
4. Consecutive terms of the series have opposite signs.
5. The sign of \( \epsilon_\mu r_\mu \) is the same for all \( \mu \) from one and the same interval \( I_q \).
All these properties are proved as in case 3.
Now let \( n_q \) be the largest integer not exceeding \( \mu_q^*/2 \). For any choice of signs
\[
\chi(z) = \sum_{n=4}^{\infty} \pm (z(1+z)^2/4)^{n_q} = \sum c_n z^n
\]
is regular in the domain \( |z(1+z)^2| < 4 \) which contains \( |z| \leq 1, z \neq 1 \).
The coefficient \( c_n \) is 0, whenever \( n \) is not in one of the intervals

\[\text{License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use}\]
\( n_q \leq n \leq 3n_q \). By (5) and (6) this implies that \( c_n = 0 \) for \( n \in I_q \) \( (q = 1, 2, \ldots) \). All the nonzero \( c_n \) whose indices are in a fixed interval \( I_q \) have the same sign. As \( n \to \infty \) through the sequence \( \mu_1^*, \mu_2^*, \ldots \), \( \left| c_n \right|^{1/n} \to 1 \). This shows that \( \chi(z) \) has radius of convergence 1 and so \( z = 1 \) is the only singularity of \( \chi(z) \) on the unit circle. The function formed by Hadamard multiplication of \( \chi(z) \) and \( \psi(z) \), i.e. \( \sum_{\mu \in M} r_\mu c_{\mu - 1/2} z_\mu^{-1/2} = f(z) \) is therefore analytic at all points of \( |z| = 1 \) at which \( \psi \) is analytic, i.e. on \( |z| = 1, x > 0 \). If the \( \pm \) signs in the definition of \( \chi(z) \) are chosen suitably, it follows from property 5 of \( \psi(z) \) that the coefficient of \( z^{-1/2} \) in \( f(z) \) and \( c_{-1/2} \) are of the same sign, since the power series of \( f(z) \) contains no terms whose indices are not in one of the intervals \( I_q \). By adding an entire function to \( f(z) \), we can form a power series without vanishing terms which satisfies all requirements.

4. The following example shows that there are sequences \( \{ \epsilon_k \} \) such that every closed semi-circle on the circle of convergence of \( \sum \epsilon_k a_k z^k \) \( (a_k > 0) \) contains at least one singularity.

Let
\[
\epsilon_k = 1, \quad k \equiv 0, 1 \pmod{4}, \\
\epsilon_k = -1, \quad k \equiv 2, 3 \pmod{4}.
\]

Then \( F(z) = \sum \epsilon_k a_k z^k \) \( (a_k > 0) \), \( \lim\sup a_k^{1/k} = 1 \) is a power series whose sequence of coefficients has sign-changes with density 1/2. By a theorem of Pólya (see [1, p. 51]) this implies that \( F(z) \) has a singularity on \( |z| = 1, |\arg z| \leq \pi/2 \). But \( F(-z) \) is again a function whose power series coefficients have sign-changes of density 1/2. Therefore Pólya's theorem shows that \( F(z) \) has a singularity on \( |z| = 1, |\arg (-z)| \leq \pi/2 \). Since \( F \) has real coefficients, the singularities of \( F \) are symmetrically situated with respect to the real axis. It is now easy to see that \( F \) has singularities on every closed semi-circle on \( |z| = 1 \).

REFERENCES


CORNELL UNIVERSITY