A THEOREM ON POWER SERIES WHOSE COEFFICIENTS HAVE GIVEN SIGNS

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1. The following theorem, first proved by A. Hurwitz and G. Pólya, is well known ([3] or [1, p. 99]).

If \( \sum_{k=0}^{\infty} a_k z^k \) is a power series of finite radius of convergence, then it is possible to find a sequence \( \{ \epsilon_k \} \) (\( \epsilon_k = \pm 1 \)) such that the series \( \sum_{k=0}^{\infty} \epsilon_k a_k z^k \) has the circle of convergence as natural boundary.

In this note I prove the following companion-piece to Pólya's theorem.

**Theorem.** If \( \{ \epsilon_k \} \sum_{k=0}^{\infty} \) is a sequence with \( \epsilon_k = \pm 1 \), then there is always a power series \( \sum a_k z^k \), \( a_k > 0 \), of finite radius of convergence such that the series \( \sum \epsilon_k a_k z^k \) can be analytically continued across a semi-circle on its circle of convergence.

This theorem answers in the negative the question: Is there a "universal scrambling sequence" \( \{ \epsilon_k \} \), \( \epsilon_k = \pm 1 \), turning every power series \( \sum a_k z^k \) with positive coefficients into a power series \( \sum \epsilon_k a_k z^k \) having the circle of convergence as natural boundary? This problem was raised by Mrs. Turán, and I am indebted to Dr. P. Erdős for communicating it to me.

An example (§4) shows that the semi-circle in the statement of the theorem can not be replaced by a larger arc.

A question which remains open is to find a corresponding theorem for the case that \( \{ \epsilon_k \} \) is a given sequence of complex numbers of absolute value one.

2. The following lemmas are required.

**Lemma 1.** Let \( \Lambda = \{ \lambda_n \} \) be a sequence of positive numbers no two of which are at a distance less than \( c > 0 \) from each other. Let

\[
g(z) = \prod_{\lambda \in \Lambda} \frac{\lambda - z}{\lambda + z} e^{2z/\lambda}.
\]

Then there are constants \( A \) and \( B \) such that in \( x \geq 0 \), \( |z - \lambda| \geq c/4(\lambda \in \Delta) \)

\[
0 < (Be^{\phi(r)})^2 \leq |g(z)| \leq (Ae^{\phi(r)})^2
\]

where \( r = |z| = |x + iy| \) and

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\[
\phi(r) = \sum_{\lambda < r; \lambda \in \Lambda} 2/\lambda.
\]

For a proof of this lemma see [2, Lemmas 3 and 4].

**Lemma 2.** Let \( M = \{\mu\} \) be a sequence of positive numbers whose mutual distances are \( \geq 1 \). Suppose that the function \( h(\xi) = h(\xi + i\eta) \) is regular in the region \( \xi \geq 0, \xi \neq 0 \) except for simple poles at the points \( \xi = \mu \in M \). Suppose further that there are positive constants \( A, \alpha, \beta \) (\( \beta < \pi \)) such that

\[
|h(\xi)| = |h(\xi + i\eta)| < Ae^{-\alpha|\eta|}
\]

in \( \xi \geq 0, \xi \neq 0 \), except in circles of radius 1/4 with centers at the points \( \mu \in M \).

Then the function

\[
H(z) = \sum_{\mu \in M} r_\mu z^\mu
\]

is regular in the sector \( 0 < |z|, |\text{arg} z| < \beta \); where \( r_\mu \) is the residue of \( h(\xi) \) at \( \mu \).

**Proof.** Let \( C_R \) be the semicircle \( |\xi| = R, \xi \geq 0 \) and let \( L_R \) be a curve with endpoints \( \xi = iR \) and \( \xi = -iR \) which runs along the imaginary axis except for an indentation into the right half plane near \( \xi = 0 \).

By the residue theorem

\[
\frac{1}{2\pi i} \int h(\xi) e^{-k\xi} d\xi = \sum_{\mu < R; \mu \in M} r_\mu e^{-k\mu}
\]

where the integration is along \( C_R + L_R \). If the number \( k \) is chosen positive and larger than \( \alpha \), then on \( C_R \)

\[
|h(\xi)e^{-k\xi}| < Ae^{(\alpha-k)|\xi|},
\]

and therefore

\[
\int_{C_R} h(\xi)e^{-k\xi} d\xi \to 0
\]

as \( R \to \infty \) through a sequence of values avoiding the intervals \( |\xi - \mu| < 1/4 \) on the real axis. It follows that for \( z = e^{-k} (k \geq \alpha) \) the series

\[
\sum_{\mu \in M} r_\mu z^\mu
\]

converges to a function
(2) \[ H(z) = \sum r_m z^m = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} h(\zeta) z^d d\zeta, \]

where the path of integration is the imaginary axis with an indentation near \( \zeta = 0 \). For purely imaginary values of \( \zeta, |z^i| = |\zeta^i| = e^{-|\arg \zeta|}. \) Hence, \( |\arg z| \leq \beta < \beta' \)\( |h(i\eta)z^i| < A e^{-\beta|\eta|+|\zeta| \cdot |\arg z|} < A e^{-(\beta-\beta')|\eta|}. \) This shows that the integral on the right-hand side of (2) is uniformly convergent in \( |\arg z| \leq \beta' < \beta. \) Therefore it defines the analytic continuation of \( H(z) \) into the whole sector \( |\arg z| < \beta. \)

3. PROOF OF THE THEOREM. Let \{\lambda\} = \Lambda be the set of those odd multiples of 1/2 for which \( \epsilon_{\lambda-1/2} \epsilon_{\lambda+1/2} = -1. \) Write

\[ \phi(r) = \sum_{\lambda < r} 2/\lambda, \]
\[ m = \liminf_{n \to \infty} (\phi(r) - \log r), \]
\[ M = \limsup_{r \to \infty} (\phi(r) - \log r). \]

We consider separately the five cases:

(i) \( -\infty < M < \infty. \) (ii) \( M = -\infty. \) (iii) \( -\infty < m < \infty. \)

(iv) \( m = \infty. \) (v) \( m = -\infty, M = \infty. \)

These cases are not mutually exclusive, but they cover all possibilities.

(i) \( -\infty < M < \infty. \) Define \( g(z) \) by (1). By Lemma 1

\[ 0 < \limsup_{n \to \infty} |g(n)|^{1/n} = C < \infty \quad (n = 1, 2, \ldots). \]

The function

\[ h(\zeta) = (C\zeta)^{-\pi} g(\zeta) \csc \pi \zeta \]

satisfies the hypotheses of Lemma 2 with \( \beta = \pi/2, M = \{1, 2, 3, \ldots\}. \)

The residue of \( h(\zeta) \) at \( n \) is

\[ r_n = (-1)^n g(n)/\pi(Cn)^n. \]

By the choice of \( C, \) the series

\[ \sum r_n(-z)^n = \sum c_n z^n = f(z) \]

has radius of convergence 1. The sign of \( c_n \) is the same as that of \( g(n). \) But \( g(x) \) changes sign between those integers \( k, k+1 \) for which \( \epsilon_k \) and \( \epsilon_{k+1} \) are of opposite sign and nowhere else. Hence \( \epsilon_n g(n) \) is of constant sign. Also, by Lemma 2, \( f(z) \) is regular in \( |\arg (-z)| < \pi/2, \)
i.e. in $x<0$. Therefore one of the two functions $\pm f(z)$ has the required properties.

(ii) $M = -\infty$. We can find a sequence $\{\nu\}$ of odd multiples of $1/2$ which has no terms in common with $\Lambda$ and for which \( \lim \sup \{\phi(r) + \sum_{r<r} 4/\nu - \log r\} = 0 \), say. The construction of the previous case can now be used, if $g(\xi)$ is replaced by

\[
g(\xi) \cdot \left\{ \prod_{\nu} \frac{\nu - \xi}{\nu + \xi} e^{2\xi/\nu} \right\}^2.
\]

(iii) $-\infty < m < \infty$. Let $g(\xi)$ again be defined by (1). If $D$ is any positive number, the function

\[
h(\xi) = (D\xi)^{\beta}/g(\xi)
\]

satisfies the hypotheses of Lemma 2, with $\{\mu\} = \{\lambda\}$, $\beta = \pi/2$. The residue at $\lambda = \rho$ is

\[
r_\rho = (D\rho)^\beta/g'(\rho).
\]

Now

\[
g'(\rho) = \prod_{\lambda \in \Lambda, \lambda \neq \rho} \frac{\lambda - \rho}{\lambda + \rho} e^{2\rho/\lambda} \cdot \frac{e^{2}}{2\rho}
\]

and so, by Lemma 1 $g'(\rho)$ lies between

\[
(B_1 e^{\phi(\rho)})^\rho \text{ and } (A_1 e^{\phi(\rho)})^\rho,
\]

where $A_1$ and $B_1$ are independent of $\rho$. Since $\lim \inf (\phi(\rho) - \log \rho)$ is finite, the constant $D$ can be adjusted so that the series

\[
\sum_{\rho \in \Lambda} r_\rho \cdot z^{\rho-1/2} = \psi(z)
\]

has radius of convergence 1. By Lemma 2 $\psi(z)$ is regular in $x > 0$. The values of $\epsilon$ at successive terms of the sequence $\{\rho - 1/2\}$ are of opposite sign, since two such integers are separated by exactly one term of the sequence $\Lambda$.

The coefficients $r_\rho$ of two consecutive terms in the power series are also of opposite sign, since the slope of $g(x)$ has opposite signs at successive zeros $\rho$ of $g(x)$.

Therefore $\pm \psi(z)$ satisfies all requirements, except that it has zero coefficients for all integers which are not of the form $\rho - 1/2$, $\rho \in \Lambda$. By adding to $\pm \psi(z)$ an entire function whose power series has suitable signs, an example satisfying all requirements is obtained.

\* This is easily verified, if the inequality $\theta \sin \theta \leq (\pi/2)(1 - \cos \theta)$ is noted.
(iv) $m = \infty$. We can choose a sub-sequence of $\Lambda$ such that we have case (iii) for the new sequence and such that any two consecutive terms of the sub-sequence are separated by an even number of terms in the original sequence $\Lambda$. The construction of case 3 can now be applied.

(v) $m = -\infty$, $M = \infty$. Put

$$\gamma(r) = \phi(r) - \log r = \sum_{\lambda<r} 2/\lambda - \log r.$$ 

The hypothesis $m = -\infty$, $M = \infty$ implies that given $q > 0$ there are arbitrarily large numbers $h, k$ such that

$$\inf_{h<r<k} \gamma(r) - \gamma(h) = -q$$

and

$$0 \leq \gamma(k) - \gamma(h) \leq 1.$$

A set $M = \{\mu\}$ is now constructed as follows.

First a sequence of nonoverlapping intervals $I_1, I_2, \cdots$ is chosen such that in $I_q = (h_q, k_q)$ (3) and (4) hold, with $h = h_q$ and $k = k_q$, and such that $h_{q+1} > 10k_q$ $(q = 1, 2, \cdots)$. All terms of $\Lambda$ which lie in $I_q$ are terms of the new sequence $M$, in fact $M \cap I_q = \Lambda \cap I_q$ $(q = 1, 2, \cdots)$.

If $\delta(r) = \sum_{\mu<r, \mu \in M} 2/\mu - \log r$, then it follows that (3) and (4) hold with $\delta$ in place of $\gamma$ and $h = h_q$, $k = k_q$.

Next we select further positive odd multiples of $1/2$ as members of $M$.

At first enough of these are chosen from the interval $(0, h_1)$ to make

$$0 < \delta(h_1) \leq 1.$$ 

This gives, by (4),

$$0 \leq \delta(k_1) \leq 2.$$ 

Next enough terms are added between $k_1$ and $h_2$ to make

$$0 \leq \delta(r) \leq 2 \quad (k_1 \leq r \leq h_2)$$

and

$$1 \leq \delta(h_2) \leq 2.$$ 

Then, by (3)

$$1 \leq \delta(k_2) \leq 3$$

and we make

$$1 \leq \delta(r) \leq 3 \quad (k_2 \leq r \leq h_3),$$

$$2 \leq \delta(h_3) \leq 3.$$
Continuing in this way we can choose $M$ so that
\[ q - 2 \leq \delta(r) \leq q \quad (k_{q-1} \leq r \leq k_q; q = 2, 3, \ldots). \]
Hence $\delta(r) \to \infty$ as $r \to \infty$ through any set of values outside the intervals $I_q$. But by (3)
\[ -1 \leq \lim \inf \delta(r) \leq 0. \]
Since $\delta(r)$ has local minima at the $r \in M$, there is a $\mu = \mu_q^* \in I_q$ such that
\[ -1 \leq \delta(\mu_q^*) \leq 0; \quad \lim \inf \delta(r) = \lim \inf \delta(\mu_q^*). \]
Since
\[
\delta(h_q) - \delta(\mu_q^*) = \log \mu_q^*/h_q - \sum_{h_q \leq \mu < \mu_q^*} 2/\mu \geq q - 2
\]
(5) \[ \mu_q^* > 3h_q \quad (q \geq 4). \]
Similarly
(6) \[ \mu_q^* < k_q/3 \quad (q \geq 4). \]
Now let $g_1(z)$ be defined by (1) with $M$ in place of $A$. As in case (iii) we derive a function
\[ \psi(z) = \sum_{\mu \in M} r_{\mu} \mu^{1/2} \]
from the auxiliary function $h(\xi) = (D\xi)^{\xi}/g(\xi)$. This function $\psi(z)$ has the following properties:
1. Its power series has radius of convergence 1.
2. As $\mu \to \infty$ through the sequence $\mu_1^*, \mu_2^*, \ldots$,
\[ \lim \sup |r_{\mu}|^{1/(\mu-1/2)} = 1. \]
3. $\psi(z)$ is regular in $x > 0$.
4. Consecutive terms of the series have opposite signs.
5. The sign of $\epsilon_{\mu} r_{\mu}$ is the same for all $\mu$ from one and the same interval $I_q$.
All these properties are proved as in case 3.
Now let $n_q$ be the largest integer not exceeding $\mu_q^*/2$. For any choice of signs
\[ \chi(z) = \sum_{q=4}^{\infty} \pm (z(1 + z)^2/4)^{n_q} = \sum c_n z^n \]
is regular in the domain $|z(1+z)^2| < 4$ which contains $|z| \leq 1$, $z \neq 1$.
The coefficient $c_n$ is 0, whenever $n$ is not in one of the intervals.
\( n_q \leq n \leq 3n_q \). By (5) and (6) this implies that \( c_n = 0 \) for \( n \in I_q \) \((q = 1, 2, \ldots)\). All the nonzero \( c_n \) whose indices are in a fixed interval \( I_q \) have the same sign. As \( n \to \infty \) through the sequence \( \mu_1, \mu_2, \ldots \), \( |c_n|^{1/n} \to 1 \). This shows that \( \chi(z) \) has radius of convergence \( 1 \) and so \( z = 1 \) is the only singularity of \( \chi(z) \) on the unit circle. The function formed by Hadamard multiplication of \( \chi(z) \) and \( \psi(z) \), i.e. \( \sum_{\mu \in M} r_{\mu} c_{\mu-1/2} z^{\mu-1/2} = f(z) \) is therefore analytic at all points of \( |z| = 1 \) at which \( \psi \) is analytic, i.e. on \( |z| = 1, x > 0 \). If the \( \pm \) signs in the definition of \( \chi(z) \) are chosen suitably, it follows from property 5 of \( \psi(z) \) that the coefficient of \( z^{\mu-1/2} \) in \( f(z) \) and \( e^{\mu-1/2} \) are of the same sign, since the power series of \( f(z) \) contains no terms whose indices are not in one of the intervals \( I_q \). By adding an entire function to \( f(z) \), we can form a power series without vanishing terms which satisfies all requirements.

4. The following example shows that there are sequences \( \{ \epsilon_k \} \) such that every closed semi-circle on the circle of convergence of \( \sum \epsilon_k a_k z^k \) \((a_k > 0)\) contains at least one singularity.

Let

\[
\epsilon_k = \begin{cases} 
1, & k \equiv 0, 1 \pmod{4}, \\
-1, & k \equiv 2, 3 \pmod{4}.
\end{cases}
\]

Then \( F(z) = \sum \epsilon_k a_k z^k \) \((a_k > 0, \lim sup a_k^{1/k} = 1)\) is a power series whose sequence of coefficients has sign-changes with density \( 1/2 \). By a theorem of Pólya (see [1, p. 51]) this implies that \( F(z) \) has a singularity on \( |z| = 1, |\arg z| \leq \pi/2 \). But \( F(-z) \) is again a function whose power series coefficients have sign-changes of density \( 1/2 \). Therefore Pólya’s theorem shows that \( F(z) \) has a singularity on \( |z| = 1, |\arg (-z)| \leq \pi/2 \). Since \( F \) has real coefficients, the singularities of \( F \) are symmetrically situated with respect to the real axis. It is now easy to see that \( F \) has singularities on every closed semi-circle on \( |z| = 1 \).

REFERENCES


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