(ii) The intersection of the closed exteriors of the circles of curvature of C, and the intersection of the closed exteriors of the minimal circumscribed circles to C.

Reference


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HOMOTOPY GROUPS OF ONE-DIMENSIONAL SPACES

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In this paper we prove the following theorem:

*If S is a one-dimensional separable metric space, then \( \pi_k(S) = 0 \) for all \( k > 1 \).*

Actually it is proved that a much broader class of spaces than spheres have the property that mappings of these spaces into one-dimensional spaces are homotopic to constant maps. This class of spaces includes, for example, projective spaces and Lens spaces.

**Lemma 1.**\(^2\) Let \( X \) be a compact metric space whose one-dimensional integral singular homology group is a torsion group. Then for any finite covering \( G \) of order one by arcwise-connected open sets, \( G \) does not contain a simple loop.

**Proof.** By a simple loop we mean a simple chain such that the first and last sets are the same. Let \( K \) be the nerve of \( G \). Since \( K \) is one-dimensional, a simple loop in \( G \) implies a nonbounding one cycle in \( K \). Hence it suffices to show that \( H_1(K) = 0 \).

Let \( \phi: X \to K \) be a canonical map. For each vertex \( v \) in \( K \) we choose a point \( \psi(v) \) in the element of \( G \) corresponding to \( v \). For each edge \( \sigma \) with vertices \( v_1 \) and \( v_2 \) we extend \( \psi \) on \( \{v_1, v_2\} \) to a mapping of \( \sigma \) into the union of the two elements of \( G \) corresponding to \( v_1 \) and \( v_2 \). This is possible, since these two elements of \( G \) are arcwise connected and

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\(^2\) The method of proof used below, which extended this lemma from manifolds \( M \) with \( H_1(M) = 0 \) to compact spaces \( X \) with \( H_1(X) \) a torsion group, was suggested by the referee.
must have a non-null intersection by the definition of the nerve of a covering. This defines a map \( \psi: K \to X \), and it is easy to check that the map \( \phi \psi: K \to K \) is star related with the identity so that \( \phi \psi \) is homotopic to the identity map. It follows from this that \( \psi_*: H_1(K) \to H_1(X) \) is an isomorphism into. Since \( H_1(X) \) is a torsion group, so is \( H_1(K) \). Since \( K \) is a graph, \( H_1(K) \) is free abelian, so that \( H_1(K) = 0 \). This proves the lemma.

**Lemma 2.** If \( X \) is also a locally connected continuum, \( Y = f(X) \) is one-dimensional and \( f \) is monotone, then \( Y \) is a dendrite.

**Proof.** A dendrite is a locally connected continuum which does not contain a simple closed curve. We suppose that \( Y \) contains a simple closed curve \( \Gamma \) and obtain a contradiction.

There exists a positive number \( \epsilon \) such that from any covering of \( \Gamma \) by open sets of diameter less than \( \epsilon \) one can extract a simple loop of open sets (which may not cover \( \Gamma \)). There exists a covering \( \mathcal{U} \) of \( Y \) of order one by open sets of diameter less than \( \epsilon \). The set of all components of members of \( \mathcal{U} \) has a finite subset \( \mathcal{V} \) which is a covering of \( Y \) of order one by connected open sets of diameter less than \( \epsilon \). It follows that \( \mathcal{V} \) contains a simple loop \( V_1, \ldots, V_n \).

The covering \( G = \{ f^{-1}(V) \mid V \in \mathcal{V} \} \) is a covering of \( X \) of order one by connected open sets, and it contains a simple loop. By Lemma 1 this is not possible and Lemma 2 is proved.

**Theorem.** Let \( X \) be a locally connected continuum whose one-dimensional integral singular homology group is a torsion group. Let \( S \) be a one-dimensional separable metric space. Then any map \( f: X \to S \) is homotopic to a constant map.

**Proof.** Let \( f = gh \) be the monotone-light factorization of \( f \), and let \( Y = h(X) \). Since \( g \) is light, \( Y \) must be one-dimensional [3, p. 91]. By Lemma 2, \( Y \) is a dendrite and one of the authors [2] has shown that a dendrite is contractible. Hence \( h \) is homotopic to a constant and so is \( f \). This proves the theorem.

**Corollary.** If \( S \) is a one-dimensional separable metric space, then \( \pi_k(S) = 0 \) for \( k > 1 \).

**Remark.** If an arcwise connected one-dimensional separable metric space \( X \) has \( \pi_1(X) \) a free finitely-generated group, then all singular homology groups \( H_k(X, \mathbb{Z}) = 0 \), \( k > 1 \). This follows because \( X \) is aspherical and an aspherical space with the same fundamental group is obtained by taking a finite number of circles with one common
point. Since the homology groups of this model are trivial in dimensions higher than one, the same is true of $X$ (see [1, p. 481]).

**Remark.** For $k = 2$ this corollary can be proved without the theorem since a monotone image of the 2-sphere is a cactoid [4] which is either a dendrite or contains a 2-sphere. This also shows that if $\pi_2(S) \neq 0$, then there exists a light map of $S^2$ into $S$. However, there exists a space $S$ for which $\pi_2(S) \neq 0$ such that any light map of $S^2$ into $S$ is inessential.

**Problem.** Is every monotone image of $S^k$, $k > 1$, simply connected? For $k = 2$ the answer is "yes" by the theorem of R. L. Moore [4], but we do not know the answer for $k = 3$.

**Bibliography**


*Added in proof. This problem has been settled in the negative by R. H. Bing for $k = 3$. 

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