with mild restrictions on \( \phi \) ensures that \( u/v \) satisfies the maximum principle; and this is the property which underlies the present analysis.

**Bibliography**


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**A NOTE ON LINEAR ORDINARY DIFFERENTIAL EQUATIONS**

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Let

\[
\frac{dx}{dt} = A(t)x,
\]

where \( x \) is an \( n \)-column vector

\[
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix}
\]

and \( A = (a_{ij}(t)) \) where \( a_{ij}(t) \) are continuous real valued functions of time \((-\infty < t < +\infty)\). Let \( y^1(t), \cdots, y^n(t) \) be any \( n \)-linearly independent solutions of (1) defined for all \( t \). Let \( B^1(t), \cdots, B^n(t) \) be the \( n \) normal-orthogonal vectors obtained from the set \( \{y^i\} \) by the Gram-Schmidt orthogonalization process. Let \( B(t) \) be the orthogonal matrix whose \( j \)th column is \( B^j(t) \), and introduce a new variable \( u \) (an \( n \)-column vector) defined by

\[(2) \quad x = B(t)u.\]

\( u \) satisfies the linear differential equation

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\[
\frac{du}{dt} = C(t)u
\]

where \( C \) is related to \( A \) and \( B \) by
\[
C(t) = \begin{pmatrix} B^{-1}(t) A(t) B(t) \end{pmatrix} - B^{-1}(t) \frac{d}{dt} B(t).
\]

We have shown\(^2\) that
\[
c_{ij}(t) = 0 \quad \text{if } i > j.
\]

We propose to show that the \( c_{ij} \) satisfies certain simple formulas for \( i \leq j \), and these will imply that the \( c_{ij} \) are bounded if the \( a_{ij} \) are bounded. Our first proof\(^2\) of this fact was unsatisfactory.

We reemploy the convention that if \( B \) is an \( n \times n \) matrix, \( B_i \) will denote the \( i \)th row of \( B \) and also the row vector determined by the \( i \)th row of \( B \); \( B^i \) will denote the \( j \)th column of \( B \) and also the column vector determined by the \( j \)th column of \( B \). If \( E, F, \) and \( G \) are three matrices \((n \times n)\) and \( E = FG \), then \( E_i = F_i G \) and \( E^i = F_i G^i \), where in the latter two formulas one has the appropriate vector-matrix and matrix-vector multiplication. \( E^i_i \) will denote the \((i - j)\)th element of \( E \), and if \( E = FG \), then \( E^i_i = F^i_i G^i_i \) where the right side is scalar multiplication (of a row vector times a column vector). If \( E = (e_{ij}) \), then \( E^i_i = e_{ij} \).

From (4) one finds
\[
\frac{dB}{dt} = B_i^{-1} A B^i - B_i^{-1} \left( \frac{dB}{dt} \right)^i,
\]

or
\[
\frac{c_{ij}}{dt} = B^i_i A B^i - B^i_i \left( \frac{dB}{dt} \right)^i,
\]

this last following from the fact the \( B \) is orthogonal, i.e. \( B' = B^{-1} \) and therefore \( (B^i)_i = B_i^{-1} \), but \( (B^i')_i = (B^i)' \) or \( B'i \) (in our notation). From \( \delta_{ij} = B_i^{-1} B^i = B''^i B^i \), one finds on differentiating that
\[
\frac{d}{dt} \left( \frac{d}{dt} B^i \right) B^i = -B^i \left( \frac{d}{dt} B^i \right) = -\left( \frac{dB^i}{dt} \right) B^i.
\]

For $i=j$ (7) implies that
\[
\left(\frac{d}{dt} B'^i\right) B^i = 0;
\]
therefore (6) for $i=j$ becomes
\[
(8) \quad c_{ii} = B'^i A B^i.
\]
Formula (5) implies for $r>s$, $c_{rs}=0$; hence using (6) that
\[
B'^r \left(\frac{dB^r}{dt}\right) = B'^r A B^r, \quad r > s
\]
and this combined with (7) implies
\[
(9) \quad B'^s \left(\frac{dB^s}{dt}\right) = -B'^r A B^r \quad (r > s).
\]
For $s=i$ and $r=j$ and $i<j$ (9) substituted into (6) yields
\[
(10) \quad c_{ij} = B'^i A B^i + B'^i A B^i.
\]
Observing, when $A'=$ transpose of $A$, that $B'^i A B^i = B'^i A' B^i$ one may rewrite (10) as
\[
(11) \quad c_{ij} = B'^i (A + A') B^i.
\]

Remarks. The fact that one has “explicit” formulas for $c_{ij}$ (i.e. (5), (8), and (10)) does not appear to simplify our treatment (loc. cit.) of the theory of “generalized characteristic exponents.” Formulas (8) and (10) can of course be used to establish a number of “stability” theorems; and all such results, including the formulas themselves, carry over directly to systems of linear ordinary differential equations in Hilbert space. It is to be noted that our expression for $C$ does not depend on the derivatives of the $b_{ij}(t)$.

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